APPENDIX B

Frequently Encountered Probability Distributions

There are a number of probability distributions that occur quite frequently in the application of probability theory to practical problems. Mathematical expressions for the most common of these distributions are collected here along with their most important parameters.

The following notation is used throughout:

$\Pr(x)$—probability of the event x occurring.
$f_X(x)$—probability density function of the random variable X at the point x.
$\overline{X} = E\{X\}$—mean of the random variable X.
$\sigma_X^2 = E\{(X - \overline{X})^2\}$—variance of the random variable X.
$\phi(u) = \int_{-\infty}^{\infty} f_X(x)e^{jux}\,dx$—characteristic function of the random variable X.

Discrete Probability Functions

Bernoulli (Special Case of Binomial)

$$\Pr(x) = \begin{cases} p & x = 1 \\ q = 1 - p & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$0 < p < 1$$

$$f_X(x) = p\delta(x-1) + q\delta(x)$$

$$\overline{X} = \rho$$

$$\sigma_X^2 = pq$$

$$\phi(u) = 1 - p + pe^{ju}$$

Binomial

$$\Pr(x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

$$0 < p < 1 \qquad q = 1 - p \qquad n = 1, 2, \ldots$$

$$f_X(x) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \delta(x-k)$$

$$\overline{X} = np$$

$$\sigma_X^2 = npq$$

$$\phi(u) = [1 - p + pe^{ju}]^n$$

Pascal

$$\Pr(x) = \begin{cases} \binom{x-1}{n-1} p^n q^{x-n} & x = n, n+1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$0 < p < 1 \qquad q = 1 - p \qquad n = 1, 2, 3, \ldots$$

$$\overline{X} = np^{-1}$$

$$\sigma_X^2 = nqp^{-2}$$

$$\phi(u) = p^n e^{jnu}[1 - qe^{ju}]^{-n}$$

Poisson

$$\Pr(x) = \frac{a^x e^{-a}}{x!} \qquad x = 0, 1, 2, \ldots$$

$$a > 0$$

$$\overline{X} = a$$

$$\sigma_X^2 = a$$

$$\phi(u) = e^{a(e^{ju}-1)}$$

Continuous Distributions

Beta

$$f_X(x) = \begin{cases} \dfrac{(a+b-1)!}{(a-1)!(b-1)!} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$a > 0 \qquad b > 0$$

$$\overline{X} = \frac{a}{a+b}$$

$$\sigma_X^2 = \frac{ab}{(a+b)^2(a+b+1)}$$

Cauchy

$$f_X(x) = \frac{1}{\pi} \frac{a}{a^2 + (x-b)^2} \qquad -\infty < x < \infty$$

$$a > 0 \qquad -\infty < b < \infty$$

Mean and variance not defined

$$\phi(u) = e^{jbu - a|u|}$$

Chi-Square

$$f_X(x) = \begin{cases} \dfrac{1}{\Gamma(n/2)} 2^{-n/2} x^{(n/2)-1} e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$n = 1, 2, \ldots$$

$$\overline{X} = n$$

$$\sigma_X^2 = 2n$$

$$\phi(u) = (1 - 2ju)^{-n/2}$$

Erlang

$$f_X(x) = \begin{cases} \dfrac{a^n x^{n-1} e^{-ax}}{(n-1)!} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a > 0 \qquad n = 1, 2, \ldots$$

$$\overline{X} = na^{-1}$$

$$\sigma_X^2 = na^{-2}$$

$$\phi(u) = a^n(a - ju)^{-n}$$

Exponential

$$f_X(x) = \begin{cases} ae^{-ax} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a > 0$$

$$\overline{X} = a^{-1}$$

$$\sigma_X^2 = a^{-2}$$

$$\phi(u) = a(a - ju)^{-1}$$

Gamma

$$f_X(x) = \begin{cases} \dfrac{x^a e^{-x/b}}{a!b^{a+1}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a > -1 \qquad b > 0$$

$$\overline{X} = (a + 1)b$$

$$\sigma_X^2 = (a + 1)b^2$$

$$\phi(u) = (1 - jbu)^{-a-1}$$

Laplace

$$f_X(x) = \frac{a}{2}e^{-a|x-b|} \qquad -\infty < x < \infty$$

$$a > 0 \qquad -\infty < b < \infty$$

$$\overline{X} = b$$

$$\sigma_X^2 = 2a^{-2}$$

$$\phi(u) = a^2 e^{jbu}(a^2 + u^2)^{-1}$$

Log-Normal

$$f_X(x) = \begin{cases} \dfrac{\exp\{-[\ln(x-a)-b]^2/2\sigma^2\}}{\sqrt{2\pi}\,\sigma(x-a)} & x \geq a \\ 0 & \text{otherwise} \end{cases}$$

$$\sigma > 0 \qquad -\infty < a < \infty \qquad -\infty < b < \infty$$

$$\overline{X} = a + e^{b+.5\sigma^2}$$

$$\sigma_X^2 = e^{2b+\sigma^2}(e^{\sigma^2} - 1)$$

Maxwell

$$f_X(x) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\,a^3x^2e^{-a^2x^2/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a > 0$$

$$\overline{X} = \sqrt{8/\pi}\,a^{-1}$$

$$\sigma_X^2 = \left(3 - \frac{8}{\pi}\right)a^{-2}$$

Normal

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X}e^{-(x-\overline{X})^2/2\sigma_X^2} \qquad -\infty < x < \infty$$

$$\sigma_X > 0 \qquad -\infty < \overline{X} < \infty$$

$$\phi(u) = a^{ju\overline{X}-(u^2\sigma_X^2/2)}$$

Normal-Bivariate

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\overline{X}}{\sigma_X}\right)^2 + \left(\frac{y-\overline{Y}}{\sigma_Y}\right)^2 - \frac{2\rho}{\sigma_x\sigma_Y}(x-\overline{X})(y-\overline{Y})\right]\right\}$$

$$-\infty < x < \infty \qquad -\infty < y < \infty \qquad \sigma_X > 0 \qquad \sigma_Y > 0$$

$$-1 < \rho < 1$$

$$\phi(u, v) = \exp[ju\overline{X} + jv\overline{Y} - \frac{1}{2}(u^2\sigma_X^2 + 2\rho uv\sigma_X\sigma_Y + v^2\sigma_Y^2)]$$

Raleigh

$$f_X(x) = \begin{cases} \frac{x}{a^2}e^{-x^2/2a^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{X} = a\sqrt{\frac{\pi}{2}}$$

$$\sigma_X^2 = \left(2 - \frac{\pi}{2}\right)a^2$$

Uniform

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$-\infty < a < b < \infty$$

$$\overline{X} = \frac{a+b}{2}$$

$$\sigma_X^2 = \frac{(b-a)^2}{12}$$

$$\phi(u) = \frac{e^{jub} - e^{jua}}{ju(b-a)}$$

Weibull

$$f_X(x) = \begin{cases} abx^{b-1}e^{-ax^b} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

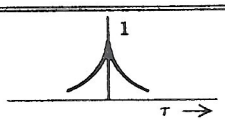
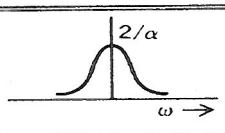
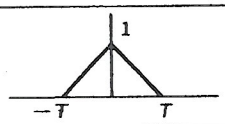
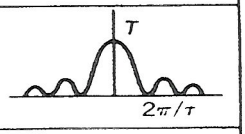
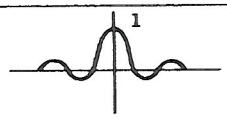
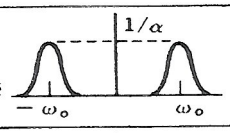
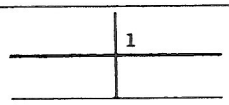
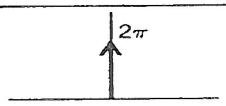
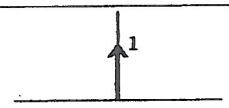
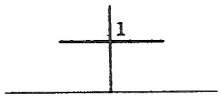
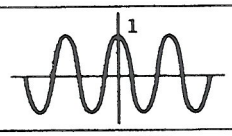
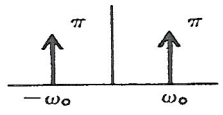
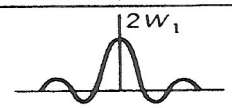
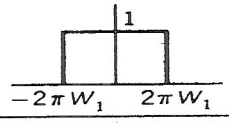
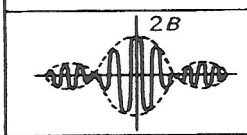
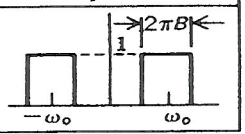
$$a > 0 \qquad b > 0$$

$$\overline{X} = \left(\frac{1}{a}\right)^{1/b}\Gamma(1 + b^{-1})$$

$$\sigma_X^2 = \left(\frac{1}{a}\right)^{2/b}\{\Gamma(1 + 2b^{-1}) - [\Gamma(1 + b^{-1})]^2\}$$

APPENDIX H

Table of Correlation Function Spectral Density Pairs

$R_X(\tau)$	$S_X(\omega)$
 $\varepsilon^{-\alpha\lvert\tau\rvert}$	$\frac{2\alpha}{\alpha^2+\omega^2}$ 
 $1-\frac{\lvert\tau\rvert}{T},\ \lvert\tau\rvert \le T$; $0,\ \lvert\tau\rvert > T$	$\frac{T\sin^2(\omega T/2)}{(\omega T/2)^2}$ 
 $\varepsilon^{-\alpha\lvert\tau\rvert}\cos\omega_o\tau$	$\frac{\alpha}{\alpha^2+(\omega-\omega_o)^2}+\frac{\alpha}{\alpha^2+(\omega+\omega_o)^2}$ 
 1	$2\pi\delta(\omega)$ 
 $\delta(\tau)$	1 
 $\cos\omega_o\tau$	$\pi\delta(\omega+\omega_o)+\pi\delta(\omega-\omega_o)$ 
 $2W_1\frac{\sin 2\pi W_1\tau}{2\pi W_1\tau}$	Low-pass Spectrum 
 $2B\frac{\sin(B\tau/2)}{(B\tau/2)}\cos\omega_o\tau$	Band-pass Spectrum 

APPENDIX A

Mathematical Tables

Table A–1 Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$$

$$\sin A - \sin B = 2 \sin \frac{1}{2}(A - B) \cos \frac{1}{2}(A + B)$$

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$$

$$\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A = \cos^2 A - \sin^2 A$$

Table 8.4-1 Input/Output Relations for WSS Sequences and Linear Systems

Random Sequence: $Y[n] = h[n] * X[n]$	Output Mean: $\mu_Y = H(0)\mu_X$
Crosscorrelations: $R_{XY}[m] = R_{XX}[m] * h^*[-m]$ $R_{YX}[m] = h[m] * R_{XX}[m]$ $R_{YY}[m] = R_{YX}[m] * h^*[-m]$	Cross–Power Spectral Densities: $S_{XY}(\omega) = S_{XX}(\omega)H^*(\omega)$ $S_{YX}(\omega) = H(\omega)S_{XX}(\omega)$ $S_{YY}(\omega) = S_{YX}(\omega)H^*(\omega)$
Autocorrelation: $R_{YY}[m] = h[m] * h^*[-m] * R_{XX}[m]$ $= g[m] * R_{XX}[m]$	Power Spectral Density: $S_{YY}(\omega) = \lvert H(\omega)\rvert^2 S_{XX}(\omega)$ $= G(\omega)\, S_{XX}(\omega)$

Output Power and Variance:

$$E\{|Y[n]|^2\} = R_{YY}[0] = \tfrac{1}{2\pi}\int_{-\pi}^{+\pi} |H(\omega)|^2 S_{XX}(\omega)d\omega$$

$$\sigma_Y^2 = R_{YY}[0] - |\mu_Y|^2$$

Table 9.5-1 Correlation Function Properties of Corresponding Power Spectral Densities

Random Process	Correlation Function	Power Spectral Density
$X(t)$	$R_{XX}(\tau)$	$S_{XX}(\omega)$
$aX(t)$	$\lvert a\rvert^2 R_{XX}(\tau)$	$\lvert a\rvert^2 S_{XX}(\omega)$
$X_1(t) + X_2(t)$ with X_1 and X_2 orthogonal	$R_{X_1X_1}(\tau) + R_{X_2X_2}(\tau)$	$S_{X_1X_1}(\omega) + S_{X_2X_2}(\omega)$
$X'(t)$	$-d^2R_{XX}(\tau)/d\tau^2$	$\omega^2 S_{XX}(\omega)$
$X^{(n)}(t)$	$(-1)^n d^{2n}R_{XX}(\tau)/d\tau^{2n}$	$\omega^{2n} S_{XX}(\omega)$
$X(t)\exp(j\omega_0 t)$	$\exp(j\omega_0\tau)R_{XX}(\tau)$	$S_{XX}(\omega - \omega_0)$
$X(t)\cos(\omega_0 t + \Theta)$ with independent Θ uniform on $[-\pi, +\pi]$	$\frac{1}{2}R_{XX}(\tau)\cos(\omega_0\tau)$	$\frac{1}{4}[S_{XX}(\omega+\omega_0) + S_{XX}(\omega-\omega_0)]$
$X(t) + b$ $(E[X(t)] = 0)$	$R_{XX}(\tau) + \lvert b\rvert^2$	$S_{XX}(\omega) + 2\pi\lvert b\rvert^2\delta(\omega)$

Table 9.5-2 Input/Output Relations for Linear Systems with WSS Inputs

WSS Random Process: $Y(t) = h(t) * X(t)$	Output Mean: $\mu_Y = H(0)\mu_X$
Crosscorrelations: $R_{XY}(\tau) = R_{XX}(\tau) * h^*(-\tau)$ $R_{YX}(\tau) = h(\tau) * R_{XX}(\tau)$ $R_{YY}(\tau) = R_{YX}(\tau) * h^*(-\tau)$	Cross–Power Spectral Densities: $S_{XY}(\omega) = S_{XX}(\omega)H^*(\omega)$ $S_{YX}(\omega) = H(\omega)S_{XX}(\omega)$ $S_{YY}(\omega) = S_{YX}(\omega)H^*(\omega)$
Autocorrelation: $R_{YY}(\tau) = h(\tau) * R_{XX}(\tau) * h^*(-\tau)$ $= g(\tau) * R_{XX}(\tau)$	Power Spectral Density: $S_{YY}(\omega) = \lvert H(\omega)\rvert^2 S_{XX}(\omega)$ $= G(\omega)S_{XX}(\omega)$

Output Power and Variance:

$$E\{|Y(t)|^2\} = R_{YY}(0) = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} |H(\omega)|^2 S_{XX}(\omega)d\omega$$

$$\sigma_Y^2 = R_{YY}(0) - |\mu_Y|^2$$