ECE 4260 Problem Set 9 Solutions

Problem 9.1 (9.2 in Stark and Woods)

(a)

$$\begin{aligned}
\mu_X(t) &\triangleq E[X(t)] \\
&= \sum_{n=-\infty}^{+\infty} E[X[n]] \frac{\sin \pi(t-nT)/T}{\pi(t-nT)/T} \\
&= \mu_X \sum_{n=-\infty}^{+\infty} 1 \frac{\sin \pi(t-nT)/T}{\pi(t-nT)/T} \\
&= \mu_X \cdot 1 \quad \text{(since } g(t)=1 \text{ is bandlimited with samples } g(nT)=1\text{)} \\
&= \mu_X.
\end{aligned}$$

(b)

$$\begin{aligned}
R_{XX}(t_1,t_2) &\triangleq E[X(t_1)X^*(t_2)] \\
&= \sum_{\text{all } m,n} R_{XX}[m-n] \text{sinc}\left(\frac{t_1-mT}{T}\right) \text{sinc}\left(\frac{t_2-nT}{T}\right),
\end{aligned}$$

where the sinc function is defined as $\text{sinc}(\tau) \triangleq \frac{\sin \pi\tau}{\pi\tau}$, so we can write the above as

$$\begin{aligned}
R_{XX}(t_1,t_2) &= R_{XX}[0] \cdot \sum_n \text{sinc}\left(\frac{t_1-nT}{T}\right) \text{sinc}\left(\frac{t_2-nT}{T}\right) \\
&\quad + R_{XX}[1] \cdot \sum_n \text{sinc}\left(\frac{t_1-(n+1)T}{T}\right) \text{sinc}\left(\frac{t_2-nT}{T}\right) + \cdots \\
&= \sum_m R_{XX}[m] r_m(t_1,t_2),
\end{aligned}$$

where $r_m(t_1,t_2) \triangleq \sum_n \text{sinc}\left(\frac{t_1-(n+m)T}{T}\right) \text{sinc}\left(\frac{t_2-nT}{T}\right)$, so then we have

$$\begin{aligned}
R_{XX}(t_1,t_2) &= \sum_m R_{XX}[m] \text{sinc}\left(\frac{(t_1-t_2)-mT}{T}\right) \quad \text{(see below)} \\
&= R_{XX}(t_1-t_2) \quad \text{and so } X(t) \text{ is WSS.}
\end{aligned}$$

To see that $r_m(t_1,t_2) = \text{sinc}\left(\frac{(t_1-t_2)-mT}{T}\right)$, we proceed as follows: Fix m and t_1 and define the function $g(t_2) \triangleq \text{sinc}\left(\frac{(t_1-t_2)-mT}{T}\right)$. Clearly g is bandlimited with coefficients $g(nT) = \text{sinc}\left(\frac{t_1-(n+m)T}{T}\right)$, so that

$$\begin{aligned} g(t_2) &= \operatorname{sinc}\left(\frac{(t_1-t_2)-mT}{T}\right) \\ &= \sum_n g(nT)\operatorname{sinc}\left(\frac{t_2-nT}{T}\right) \\ &= \sum_n \operatorname{sinc}\left(\frac{t_1-(n+m)T}{T}\right)\operatorname{sinc}\left(\frac{t_2-nT}{T}\right). \end{aligned}$$

Problem 9.2 (9.3 in Stark and Woods)

The random sequence $B[n]$ is Bernoulli and its values ± 1 occur with equal probabilities $1/2$. We have $X(t) \triangleq \sqrt{p}\sin\left(2\pi f_0 t + B[n]\frac{\pi}{2}\right)$ where $\sqrt{p}$ and f_0 are given real numbers.

(a)

$$\begin{aligned} \mu_X(t) &\triangleq E[X(t)] \\ &= E\left[\sqrt{p}\sin\left(2\pi f_0 t + B[n]\frac{\pi}{2}\right)\right] \\ &= \sqrt{p}E\left[\sin\left(2\pi f_0 t + B[n]\frac{\pi}{2}\right)\right] \\ &= \sqrt{p}\left(\frac{1}{2}\sin\left(2\pi f_0 t + \frac{\pi}{2}\right) + \frac{1}{2}\sin\left(2\pi f_0 t - \frac{\pi}{2}\right)\right) \\ &= \sqrt{p}\left(\frac{1}{2}\cos\left(2\pi f_0 t\right) + \frac{1}{2}\left(-\cos\left(2\pi f_0 t\right)\right)\right) \\ &= \sqrt{p}\left(\frac{1}{2}\cos\left(2\pi f_0 t\right) - \frac{1}{2}\cos\left(2\pi f_0 t\right)\right) \\ &= 0. \end{aligned}$$

(b) For this real-valued process, $K_{XX}(t,s) = E[X(t)X(s)]$ since the means are zero. To evaluate $E[X(t)X(s)]$, we consider two cases:

(i) Case 1: $nT \le t, s < (n+1)T$, i.e. t and s are in the same half-open interval $[nT, (n+1)T)$. Then

$$\begin{aligned} E[X(t)X(s)] &= \frac{1}{2}\sqrt{p}\sin\left(2\pi f_0 t + \frac{\pi}{2}\right)\sqrt{p}\sin\left(2\pi f_0 s + \frac{\pi}{2}\right) + \frac{1}{2}\sqrt{p}\sin\left(2\pi f_0 t - \frac{\pi}{2}\right)\sqrt{p}\sin\left(2\pi f_0 s - \frac{\pi}{2}\right) \\ &= \frac{1}{2}p\cos\left(2\pi f_0 t\right)\cos\left(2\pi f_0 s\right) + \frac{1}{2}p\cos\left(2\pi f_0 t\right)\cos\left(2\pi f_0 s\right) \\ &= p\cos\left(2\pi f_0 t\right)\cos\left(2\pi f_0 s\right). \end{aligned}$$

(ii) Case 2: $nT \le t < (n+1)T, mT \le s < (m+1)T$, with $n \ne m$, i.e. t and s are in different intervals. In this case $X(t)$ and $X(s)$ are independent, so $E[X(t)X(s)] = E[X(t)]E[X(s)]$, but here the means are zero, hence $E[X(t)X(s)] = 0$.

Combining the two cases we can write

$$\begin{aligned} K_{XX}(t,s) &= E[X(t)X(s)] \\ &= \begin{cases} p\cos(2\pi f_0 t)\cos(2\pi f_0 s), & nT \le t, s < (n+1)T \text{ for some integer } n. \\ 0, & \text{else.} \end{cases} \end{aligned}$$

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Problem 9.3 (9.25 in Stark and Woods)

(a)

$$\dot{\mu}_Y(t) + a\mu_Y(t) = \mu_X \quad \text{and} \quad h(t) = e^{-at}u(t), \quad \text{so}$$

$$\begin{aligned} \mu_Y(t) &= \mu_X \int_0^\infty e^{-at}dt \\ &= \mu_X/a. \end{aligned}$$

(b) From (9.5-5a)

$$\begin{aligned} R_{XY}(\tau) &= h^*(-\tau) * R_{XX}(\tau) \\ &= \int_{-\infty}^{\infty} h^*(-t)R_{XX}(\tau - t)dt \\ &= \int_{-\infty}^{\infty} h^*(t')R_{XX}(\tau + t')dt', \qquad \text{with } t' \triangleq -t, \\ &= \int_0^{\infty} e^{-at'}[\delta(\tau + t') + \mu_X^2]dt' \\ &= \begin{cases} e^{a\tau} + \mu_X^2/a, & \tau \le 0, \\ 0 + \mu_X^2/a, & \tau > 0, \end{cases} \\ &= e^{a\tau}u(-\tau) + \mu_X^2/a \qquad \text{for all } \tau. \end{aligned}$$

Then, from (9.5-5b),

$$\begin{aligned}
R_{YY}(\tau) &= \int_{-\infty}^{\infty} h(\tau_1) R_{XY}(\tau - \tau_1) d\tau_1 \\
&= \int_0^{\infty} e^{-a\tau_1}[e^{a(\tau-\tau_1)} + \mu_X^2/a] d\tau_1, \quad \text{for} \quad \tau < 0, \\
&= e^{a\tau} \int_0^{\infty} e^{-2a\tau_1} d\tau_1 + \mu_X^2/a, \\
&= e^{a\tau}/2a + (\mu_X/a)^2, \ \tau < 0.
\end{aligned}$$

For $\tau > 0$,

$$\begin{aligned}
R_{YY}(\tau) &= e^{a\tau} \int_{\tau}^{\infty} e^{-2a\tau_1} d\tau_1 + (\mu_X/a)^2, \\
&= e^{-a\tau}/2a + (\mu_X/a)^2, \tau < 0.
\end{aligned}$$

Thus for all τ, we have

$$R_{YY}(\tau) = e^{-a|\tau|}/2a + (\mu_X/a)^2 .$$

(c) Since $\mu_Y = \mu_X/a$, then for the covariance, we have

$$K_{YY}(\tau) = e^{-a|\tau|}/2a \quad \text{and} \quad \sigma_Y^2 = 1/2a.$$

Problem 9.4 (9.32 in Stark and Woods)

. (a) The input process $X(t)$ has constant mean 128. So $\mu_Y(t) = \mu_X H(0) = 128 \times 1 = 128$.

(b) For the covariance function,

$$K_{YY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |H(\omega)|^2 S_{X_cX_c}(\omega) e^{+j\omega\tau} d\omega, \qquad \text{(p32eq1)}$$

where $X_c \triangleq X - \mu_X$ is the centered version of X. Also

$$\begin{aligned}
|H(\omega)|^2 &= H(\omega)H^*(\omega) \\
&= \frac{1}{1+j\omega}\frac{1}{1-j\omega} \\
&= \frac{1}{1+\omega^2}.
\end{aligned}$$

The PSD $S_{X_cX_c}$ is determined as the FT of K_{XX} :

$$
\begin{aligned}
S_{X_oX_o}(\omega) &= \int_{-\infty}^{+\infty} 1000e^{-10|\tau|}e^{-j\omega\tau}d\tau \\
&= 1000\left(\int_{-\infty}^{0} e^{+(10-j\omega)\tau}d\tau + \int_{0}^{+\infty} e^{-(10+j\omega)\tau}d\tau\right) \\
&= 1000\left(\frac{1}{10-j\omega} + \frac{1}{10+j\omega}\right) \\
&= \frac{20,000}{100+\omega^2}.
\end{aligned}
$$

Now, we can plug into (p32eq1) to obtain

$$
\begin{aligned}
K_{YY}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{1+\omega^2}\frac{20,000}{100+\omega^2}e^{+j\omega\tau}d\omega \\
&= \frac{20,000}{2\pi(99)}\int_{-\infty}^{+\infty}\left(\frac{1}{1+\omega^2} - \frac{1}{100+\omega^2}\right)e^{+j\omega\tau}d\omega \\
&= \frac{20,000}{99}\left(\frac{1}{2}e^{-|\tau|} - \frac{1}{20}e^{-10|\tau|}\right) \\
&\doteq 101.01e^{-|\tau|} - 10.10e^{-10|\tau|}.
\end{aligned}
$$