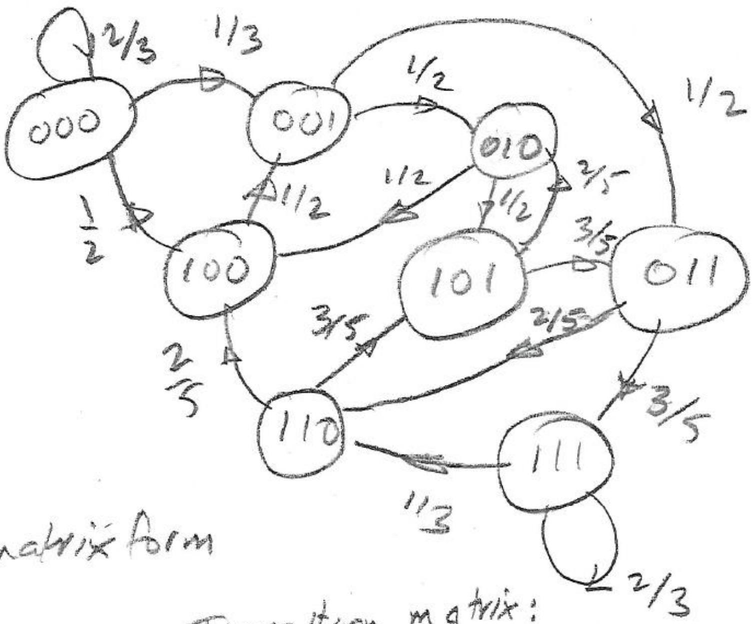
ECE 4260 Problem Set 8 Solutions

Problem 8.1

(a) I do not see any way to get by with less than 8 states:



(b) In matrix form

State	
000	1
001	2
010	3
011	4
100	5
101	6
110	7
111	8

Transition matrix:

$$\begin{bmatrix} 2/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2/5 & 3/5 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/5 & 3/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/5 & 3/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 2/3 \end{bmatrix} = A$$

$A^T \underline{P}_k = \underline{P}_{k+1}$

FOR LIMITING STATE, $A^T \underline{P} = \underline{P}$ or $(A^T - I)\underline{P} = \underline{0}$

Idea: $p_1 = \frac{2}{3} p_1 + \frac{1}{2} p_5$

$p_2 = \frac{1}{3} p_1 + \frac{1}{2} p_5$

$p_3 = \frac{1}{2} p_2 + \frac{2}{5} p_6$

$p_4 = \frac{1}{2} p_2 + \frac{3}{5} p_6$

$p_5 = \frac{1}{2} p_3 + \frac{2}{5} p_7$

$p_6 = \frac{1}{2} p_3 + \frac{3}{5} p_7$

$p_7 = \frac{2}{5} p_4 + \frac{1}{3} p_8$

$p_8 = \frac{3}{5} p_4 + \frac{2}{3} p_8$

we also have $p_1 + p_2 \cdots p_8 = 1$.

The eq's to the left are not enough. $A^T - I$ is singular. We replace any row with $(1, \cdots 1)$ and adjust the RHS.

If we elimate the last eq & substitute $p_1 + \cdots + p_8 = 1$, with a RHS of $[0, \cdots 0, 1]^T$, we can solve the eq's. (Alternatively, since A is sparse, we could solve by hand.)

Solution: $\underline{P}^T = [.1429, .0952, .0952, .1190, .0952, .1190, .1190, .2143]$

$P(\text{success}) = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{3}\right) \underline{P}$

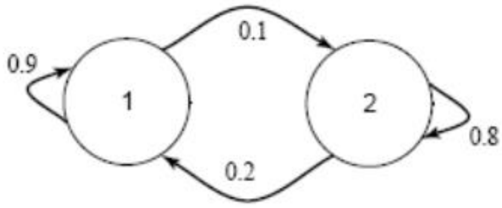
$= .5452$

Problem 8.2 (8.39 in Stark and Woods)

(a) With the two states $X = 1, 2$, we have state probability vector $\mathbf{p}$ at time n (≥ 0) given as

$$
\begin{aligned}
\mathbf{p}[n] &= (P[X[n]=1, P[X[n]=2]) \\
&= (P[X[n-1]=1]p_{11} + P[X[n-1]=2]p_{21}, P[X[n-1]=1]p_{12} + P[X[n-1]=2]p_{22}) \\
&= \mathbf{p}[n-1]\mathbf{P}, \quad \text{with } \mathbf{P} \text{ the state-transition matrix,} \\
&= \mathbf{p}[n-2]\mathbf{P}^2, \\
&\vdots \\
&= \mathbf{p}[0]\mathbf{P}^n.
\end{aligned}
$$

(b)



(c) Let p be the probability of the event {*first* transition to state 2 occuring at time n}. Then, given $X[0] = 1$, we have

$$
\begin{aligned}
p &= p_{11}^{n-1} p_{12} \\
&= (0.9)^{n-1}(0.1) \\
&= (0.1)(0.9)^{n-1}.
\end{aligned}
$$

Problem 8.3 (8.40 in Stark and Woods)

(a)

$$H(\omega) = \frac{1}{1 - re^{-j\omega}} \quad \text{and} \quad h[n] = r^n u[n],$$

so

$$
\begin{aligned}
S_{XX}(\omega) &= |H(\omega)|^2 S_{ZZ}(\omega) \\
&= \frac{1}{|1 - re^{-j\omega}|^2}\sigma_Z^2 \\
&= \frac{\sigma_Z^2}{1 + r^2 - 2r\cos\omega}.
\end{aligned}
$$

(b) We know $R_{XX}[m] = (h[m] * h^*[-m]) * \sigma_Z^2 \delta[m]$. Here, we have

$$
\begin{aligned}
h[n] * h^*[-n] &= \sum_{k=-\infty}^{+\infty} h[k]h^*[-(n-k)] \\
&= \sum_{k=-\infty}^{+\infty} r^k u[k] r^{-(n-k)} u[k-n] \\
&= r^{-n} \sum_{k=0}^{+\infty} r^{2k} u[k-n] \\
&= \begin{cases} \frac{r^n}{1-r^2}, & n \geq 0, \\ \frac{r^{-n}}{1-r^2}, & n \leq 0, \end{cases} \\
&= \frac{r^{|n|}}{1-r^2}, \qquad \text{for all } n.
\end{aligned}
$$

Thus

$$
\begin{aligned}
R_{XX}[m] &= (h[m] * h^*[-m]) * \sigma_Z^2 \delta[m] \\
&= \frac{r^{|m|}}{1-r^2} * \sigma_Z^2 \delta[m] \\
&= \left(\frac{r^{|m|}}{1-r^2} * \delta[m] \right) \sigma_Z^2 \\
&= \frac{r^{|m|}}{1-r^2} \sigma_Z^2.
\end{aligned}
$$

Problem 8.4 (8.42 in Stark and Woods)

(8.42 in S&W)

(a) FOR ANY INDEPENDENT INCREMENTS PROCESS:
where $X[0] = 0$; $X[n] = \sum_{k=1}^{n} w[n]$; and $W[n]$ is
w is uncorrelated noise with $\overline{W[n]} = \bar{W}$; $\overline{W^2[n]} = \overline{W^2}$;
and $\sigma^2_{W[n]} = \sigma^2_W$:

$$E(X[n_1]) = n_1 \bar{W} = \mu_X[n]$$

Here: $\bar{W} = \frac{1}{2}(s_1 - s_2)$

(b) $R_{XX}[n_1, n_2] = E(X[n_1]X[n_2])$

Special cases: Let $n_1 = n_2$

$$R_{XX}[n_1, n_1] = \overline{X^2[n_1]} = E\left\{ \sum_{i=1}^{n_1} w(i) \sum_{j=1}^{n_1} w(j) \right\}$$

$$= \underbrace{\sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \overline{W[i]W[j]}}_{n_1^2 \text{ terms}} = \underbrace{n_1 \overline{W^2}}_{\text{when } i=j} + \underbrace{n_1(n_1-1)\bar{W}^2}_{\text{when } i \neq j}$$

$$= n_1\left(\overline{W^2} + (n_1 - 1)\bar{W}^2\right)$$

Let $n_1 \neq n_2$ $n_2 > n_1$ w/o loss of generality.

$$E(X[n_1]X[n_2]) = E\left[(X[n_1])(X[n_1] + \sum_{n=n_1+1}^{n_2} w[n])\right]$$

$$= \overline{X^2[n_1]} + \overline{X[n_1]} \cdot (n_2 - n_1)\bar{w}$$

$$= n_1\overline{w^2} + (n_1)(n_1-1)\bar{w}^2 + n_1\bar{w}(n_2-n_1)\bar{w}$$

$$= n_1\overline{w^2} + [n_1(n_1-1) + n_1(n_2-n_1)]\bar{w}^2]$$

$$= n_1\left(\overline{w^2}[\cancel{n_1} - 1 + n_2 - \cancel{n_1}]\bar{w}^2\right]$$

$$= n_1(\overline{w^2} + (n_2-1)\bar{w}^2)$$

$$\bar{w} = \frac{1}{2}(s_1 - s_2)$$

$$\overline{w^2} = \frac{1}{2}(s_1^2 + s_2^2)$$