ECE 4260 Problem Set 7 Solutions

Problem 7.1

(a) A discrete-time random process has sample functions of the form: $X[n] = A$ where A is a Gaussian random variable of mean 2 and variance 1.

4 (i) Find the mean of $X[n]$.

$$E[X[n]] = E(A) = 2$$

4 (ii) Find the power in $X[n]$.

$$\overline{X^2[n]} = \text{power} = 2^2 + 1 = 5$$

4 (iii) Find $R_X[m_1, m_2]$, the autocorrelation function of $X[n]$.

$$R_{XX}[m_1, m_2] = A^2 = 5 \quad \forall\ m_1, m_2$$

4 (iv) Is $X[n]$ deterministic or not? Justify your answer.

Yes. Make one observation of one point of a sample function. All other values of $x[n]$ a determined.

(b) $W[n]$ is zero mean discrete-time WSS white noise with spectral height of 1.

4 (i) What is its power?

$$\text{power} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,d\omega = 1$$

$W[n]$ is put through an ideal lowpass filter with gain 1 and cutoff $\pi/3$ radians. The filter's corresponding impulse response is $\frac{\sin\frac{\pi}{3}n}{\pi n}$. The output is $Y[n]$.

4 (ii) Find $R_{YY}[m]$. $\rightarrow$

$$R_{WW}[m] * \underbrace{h[m] * h[-m]}_{h[m]} = h[m] = \frac{\sin\frac{\pi}{3}m}{\pi m}$$

$R_{WW}[m] = \delta[m]$

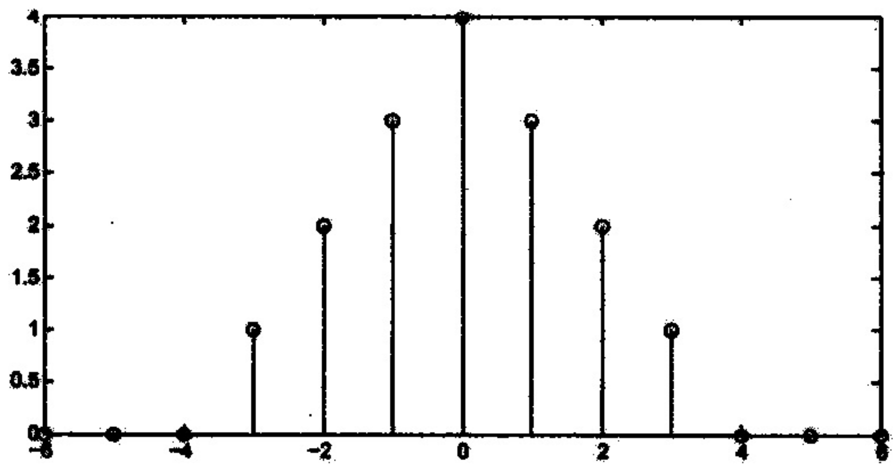
(iii) Find the variance of $Y[n]$.

$$\sigma_Y^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{YY}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} d\omega = 1/3$$

(iv) Find $R_{YW}[m]$

$$R_{YW}[m] = R_{WW}[m] * h[m] = \frac{\sin\frac{\pi}{3}m}{\pi m}$$

(c) $G[n]$ is stationary, zero mean, and has autocorrelation function as sketched below:



(i) If $G[n]$ is Gaussian, find the joint PDF for $[G[1], G[2]]$

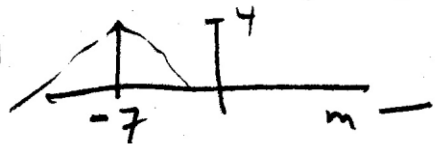
$$\begin{pmatrix} G(1) \\ G(2) \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\right]$$

(ii) Find $\frac{1}{2\pi}\int_{-\pi}^{\pi} S_G(\omega)d\omega$ where $S_G(\omega)$ is the power spectral density of $G[n]$.

$$= R_{GG}[0] = 4$$

(iii) $G[n]$ is input to a system with impulse response equal to $\delta[n-7]$ (i.e., a delay by 7). The output is $F[n]$. Find $R_{GF}[m]$. $= R_{GG}[m] * h[-m] = R_{GG}[m] * \delta[m+7]$

$= R_{GG}[m+7]$



(d) $J[n]$ and $K[n]$ are independent, zero mean stationary random processes. $R_{JJ}[m] = 2e^{-|m|}$; $R_{KK}[m] = 3e^{-(m^2)}$.

(i) Find the power in $3J[n] - 2K[m]$.

$= 9 \cdot \text{power in } J + 4 \cdot \text{power in } K$

$18 + 12 = 30$

(ii) Let $L[m] = J[m] + K[m]$. Find $R_{LJ}[m]$. $= E(L[n]\,J[n+m])$

$= E[(J[n] + K[n])(J[n+m])] = R_{JJ}[m] = 2e^{-|m|}$

(iii) $J[n]$ was obtained by passing unit spectral height white noise through a filter. Find a possible impulse response for that filter.

many ways to do this. $\Phi_{JJ}(z)$ has poles at e and e^{-1}

we only need one pole. we also need a scale.

In general, $a^{n}u[n] * a^{-n}u[-n] = \text{scale} \cdot a^{|n|}$

$\text{scale} = \sum_{n=0}^{\infty} a^{2n} = \frac{1}{1-a^2} \rightarrow h[n] = \sqrt{2}\sqrt{1-a^2}\, a^{n} u[n]$

$a = e^{-1} \quad = \sqrt{2}\sqrt{1-e^{-2}}\, e^{-n} u[n]$

is one choice

Problem 7.2 (8.22 in Stark and Woods)

Let the system be represented by operator L as $y[n] = L\{x[n]\}$. From the definition$h[n] = L\{\delta[n]\}$ with $\delta[n]$ being the discrete time impulse function $\delta[n] \triangleq \begin{cases} 1, & n = 0, \\ 0, & \text{else.} \end{cases}$ Next, using the shifting representation, we write the input sequence as $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$. Then we can compute

$$\begin{aligned} y[n] &= L\{x[n]\} \\ &= L\left\{ \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \right\} \\ &= \sum_{k=-\infty}^{+\infty} x[k]L\{\delta[n-k]\}, \qquad \text{by linearity for a continuous operator } L, \\ &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]. \\ &= x[n] * h[n]. \end{aligned}$$

Therefore $Y[n] = X[n] * h[n]$ too. Note that in order to interchange the operator L and the infinite summation operator $\sum_{k=-\infty}^{+\infty}$, we generally need that $h[n]$ be absolutely summable, i.e. $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$, a stable system. Stable operators L are *continuous* in the sense that a small change in the input sequence x results in a bounded change in the output sequence y.

(b) $A(\omega) \triangleq \sum_{n=-\infty}^{+\infty} a[n]e^{-j\omega n}$ and so,

$$\begin{aligned} a[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\omega)e^{+j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{+\infty} a[m]e^{-j\omega m} \right) e^{+j\omega n} d\omega \\ &= \sum_{n=-\infty}^{+\infty} a[m] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{+j\omega(n-m)} d\omega \right), \quad \text{by interchanging the infinite sum and the integral,} \end{aligned}$$

$$= \sum_{n=-\infty}^{+\infty} a[m]\delta[n-m]$$

$$= \quad a[n],$$

where the interchange of the infinite sum and the integral is permitted if the sequence a is absolutely summable, i.e. $\sum_{n=-\infty}^{+\infty} |a[n]| < \infty$.

(c) We have $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$, thus

$$\begin{aligned}
Y(\omega) &= \sum_{n=-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} x[k]h[n-k] \right) e^{-j\omega n} \\
&= \sum_{k=-\infty}^{+\infty} x[k] \left(\sum_{n=-\infty}^{+\infty} h[n-k]e^{-j\omega n} \right), \quad \text{by interchanging the infinite sums,} \\
&= \sum_{k=-\infty}^{+\infty} x[k]e^{-j\omega k} \left(\sum_{n=-\infty}^{+\infty} h[n-k]e^{-j\omega n}e^{+j\omega k} \right) \\
&= \sum_{k=-\infty}^{+\infty} x[k]e^{-j\omega k} \left(\sum_{n=-\infty}^{+\infty} h[n-k]e^{-j\omega(n-k)} \right) \\
&= \sum_{k=-\infty}^{+\infty} x[k]e^{-j\omega k} \left(H(\omega) \right) \\
&= H(\omega) \sum_{k=-\infty}^{+\infty} x[k]e^{-j\omega k} \\
&= H(\omega)X(\omega).
\end{aligned}$$

Note that the interchange of the infinite sums in the steps above can be justified if the infinite sum $\sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ converges uniformly. This occurs when $\sum_{k=-\infty}^{+\infty} |x[k]| \cdot |h[n-k]| < \infty$.

Problem 7.3 (8.24 in Stark and Woods)

. (a) We are given that ρ is a real constant, but in general α could be complex.

$$\begin{aligned}
K_{YY}[m] &\triangleq E[Y[n+m]Y^*[n]] \\
&= E\left[(X[n+m] - \alpha X[n+m-1])(X[n] - \alpha X[n-1])^*\right] \\
&= K_{XX}[m] - \alpha K_{XX}[m-1] - \alpha^* K_{XX}[m+1] + |\alpha|^2 K_{XX}[m] \\
&= (1+|\alpha|^2)K_{XX}[m] - \alpha K_{XX}[m-1] - \alpha^* K_{XX}[m+1] \\
&= \sigma^2 \left[(1+|\alpha|^2)\rho^{|m|} - \alpha\rho^{|m-1|} - \alpha^*\rho^{|m+1|} \right].
\end{aligned}$$

(b) To get white noise, we try α real and take $m \geq 1$, then we set

$$\begin{aligned}
0 &= (1+\alpha^2)\rho^m - \alpha\rho^{m-1} - \alpha\rho^{m+1} \\
&= \rho^m(1+\alpha^2 - \alpha/\rho - \alpha\rho) \\
&\Longrightarrow \alpha = \rho.
\end{aligned}$$

This also works, i.e. gives zero for $K_{YY}[m]$ for $m < 0$, thus $\alpha = \rho$ is a solution. The value $\alpha = \rho^{-1}$ also works to produce white noise at the system output.

(c) For $m = 0$, we then get the variance of the white noise sequence Y

$$\begin{aligned}\sigma_Y^2 &= \sigma^2(1+\alpha^2-\alpha\rho-\alpha\rho)\\ &= \sigma^2(1-\rho^2), \quad \text{with the choice } \alpha=\rho.\end{aligned}$$

Alternatively, with the choice $\alpha = \rho^{-1}$, we get $\sigma_Y^2 = (\rho^{-2}-1)$.

Problem 7.4 (8.26 in Stark and Woods)

(a) From the problem, $Y[n] = h[n] * [W[n] + X[n]]$, so

$$\begin{aligned}\mu_Y[n] &= h[n] * (\mu_W[n] + 3)\\ &= \sum_{k=0}^{\infty} \rho^k(\mu_W[n-k]+3)\\ &= \sum_{k=0}^{\infty} \rho^k(2+3)\\ &= 5\sum_{k=0}^{\infty} \rho^k\\ &= \frac{5}{1-\rho.}\end{aligned}$$

(b) The second moment of the real-valued random sequence Y is given as:

$$\begin{aligned}E[Y^2[n]] &= E\left[\left(\sum_{k=0}^{\infty} h[k](W[n-k]+3)\right)^2\right]\\ &= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l]E[(W[n-k]+3)\,(W[n-l]+3)]\\ &= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l](\sigma_W^2\delta[l-k]+4+9+6+6)\\ &= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l](\sigma_W^2\delta[l-k]+25)\\ &= \sum_{k=0}^{\infty} h^2[k]\sigma_W^2 + \sum_{(k,l)\geq 0}^{\infty} h[k]h[l](25)\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{k=0}^{\infty} h^2[k]\right)\sigma_W^2 + \left(\sum_{k=0}^{\infty} h[k]\right)^2 25 \\
&= \left(\sum_{k=0}^{\infty} \rho^{2k}\right)\sigma_W^2 + \left(\sum_{k=0}^{\infty} \rho^{k}\right)^2 25 \\
&= \frac{\sigma_W^2}{1-\rho^2} + \frac{25}{(1-\rho)^2}.
\end{aligned}
$$

(c) For the covariance function of Y, we have

$$
\begin{aligned}
K_{YY}[m,n] &= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l]K_{WW}[m-k,n-l] \\
&= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l]\sigma_W^2\delta[m-k-(n-l)] \\
&= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l]\sigma_W^2\delta[(m-n)-(k-l)] \\
&= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l]\sigma_W^2\delta[(m-n)-(k-l)] \\
&= \sum_{k=0}^{\infty} h[k]h[k-(m-n)]\sigma_W^2 \\
&= g(m-n),
\end{aligned}
$$

where $g(m) = K_{YY}[m]$, the WSS covariance function. Continuing on,

$$
\begin{aligned}
K_{YY}[m] &= \sum_{k=0}^{\infty} h[k]h[k-m]\sigma_W^2 \\
&= \sum_{k=\max(0,m)}^{\infty} \rho^k \rho^{k-m}\sigma_W^2 \\
&= \left(\sum_{k=\max(0,m)}^{\infty} \rho^{2k}\right)\rho^{-m}\sigma_W^2 \\
&= \frac{\rho^{2\max(0,m)}}{1-\rho^2}\rho^{-m}\sigma_W^2 \\
&= \rho^{|m|}\frac{\sigma_W^2}{1-\rho^2}.
\end{aligned}
$$

Thus $K_{YY}[m,n] = K_{YY}[m-n] = \rho^{|m-n|}\frac{\sigma_W^2}{1-\rho^2}$.

$$
\begin{aligned}
K_{YY}[m] &= \sum_{k=0}^{\infty} h[k]h[k-m]\sigma_W^2 \\
&= \sum_{k=\max(0,m)}^{\infty} \rho^k \rho^{k-m}\sigma_W^2 \\
&= \left(\sum_{k=\max(0,m)}^{\infty} \rho^{2k}\right)\rho^{-m}\sigma_W^2 \\
&= \frac{\rho^{2\max(0,m)}}{1-\rho^2}\rho^{-m}\sigma_W^2 \\
&= \rho^{|m|}\frac{\sigma_W^2}{1-\rho^2}.
\end{aligned}
$$

Thus $K_{YY}[m,n] = K_{YY}[m-n] = \rho^{|m-n|}\frac{\sigma_W^2}{1-\rho^2}$.

Problem 7.5 (8.32 in Stark and Woods)

We are given $R_{XX}[m] = 10e^{-\lambda_1|m|} + 5e^{-\lambda_2|m|}$ with $\lambda_1 > 0$ and $\lambda_2 > 0$. We assume $\lambda_1 \neq \lambda_2$

$$
\begin{aligned}
S_{XX}(\omega) &\triangleq \sum_{m=-\infty}^{+\infty} R_{XX}[m]e^{-j\omega m} \\
&= \sum_{m=-\infty}^{+\infty} 10e^{-\lambda_1|m|}e^{-j\omega m} + \sum_{m=-\infty}^{+\infty} 5e^{-\lambda_2|m|}e^{-j\omega m} \\
&= 10\left(\sum_{m=0}^{+\infty} e^{-\lambda_1 m}e^{-j\omega m} + \sum_{m=-\infty}^{-1} e^{+\lambda_1 m}e^{-j\omega m}\right) \\
&\quad +5\left(\sum_{m=0}^{+\infty} e^{-\lambda_2 m}e^{-j\omega m} + \sum_{m=-\infty}^{-1} e^{+\lambda_2 m}e^{-j\omega m}\right) \\
&= 10\left(\sum_{m=0}^{+\infty} e^{-(\lambda_1+j\omega)m} + \sum_{m=-\infty}^{0} e^{+(\lambda_1-j\omega)m} - 1\right) \\
&\quad +5\left(\sum_{m=0}^{+\infty} e^{-(\lambda_2+j\omega)m} + \sum_{m=-\infty}^{0} e^{+(\lambda_2-j\omega)m} - 1\right) \\
&= 10\left(\sum_{m=0}^{+\infty} e^{-(\lambda_1+j\omega)m} + \sum_{m'=0}^{+\infty} e^{-(\lambda_1-j\omega)m'} - 1\right) \\
&\quad + 5\left(\sum_{m=0}^{+\infty} e^{-(\lambda_2+j\omega)m} + \sum_{m'=0}^{+\infty} e^{-(\lambda_2-j\omega)m'} - 1\right), \quad \text{with sub } m' \triangleq -m,
\end{aligned}
$$

$$
\begin{aligned}
&= 10\left(\frac{1}{1-e^{-(\lambda_1+j\omega)}} + \frac{1}{1-e^{-(\lambda_1-j\omega)}} - 1\right) + 5\left(\frac{1}{1-e^{-(\lambda_2+j\omega)}} + \frac{1}{1-e^{-(\lambda_2-j\omega)}} - 1\right) \\
&= 10\left(\frac{1-e^{-2\lambda_1}}{1-2\cos\omega\, e^{-\lambda_1}+e^{-2\lambda_1}}\right) + 5\left(\frac{1-e^{-2\lambda_2}}{1-2\cos\omega\, e^{-\lambda_2}+e^{-2\lambda_2}}\right).
\end{aligned}
$$

Problem 7.6 (8.36 in Stark and Woods)

For this system,

$$h[n] = \frac{1}{5}\left(\delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]\right)$$

and

$$
\begin{aligned}
H(\omega) &= \frac{1}{5}(1 + 2\cos\omega + 2\cos 2\omega) \\
&= \frac{1}{5}\frac{\sin\frac{5}{2}\omega}{\sin\frac{1}{2}\omega}.
\end{aligned}
$$

Then

(a)

$$
\begin{aligned}
S_{YY}(\omega) &= |H(\omega)|^2\, S_{XX}(\omega) \\
&= \frac{1}{25}(1 + 2\cos\omega + 2\cos 2\omega)^2 \cdot 2 \\
&= \frac{2}{25}\left(\frac{\sin\frac{5}{2}\omega}{\sin\frac{1}{2}\omega}\right)^2.
\end{aligned}
$$

(b)

$$
\begin{aligned}
R_{YY}[m] &= h[m] * h[-m] * [\delta[m] \\
&= \frac{2}{25}\mathrm{triag}[m].
\end{aligned}
$$

Here, the triangular finite-support sequence triag[·] is specified as follows:

n	0	± 1	± 2	± 3	± 4	else
triag[n]	5	4	3	2	1	0