ECE 4260 Problem Set 6 Solutions

Problem 6.1

(a) $\overline{X^2[m]} = R_X[0] = \exp(-|0|/5) + 0.25 = 1.25$

(b) Power in $X[n]$ equals $\overline{X^2[0]} = 1.25$. $\sigma^2_{X[m]} = \overline{X^2[0]} - 0.25 = 1$.

(c) The diagonal elements both are equal to $\sigma^2_{X[m]} = 1$. The off-diagonal elements are equal to $e^{-\frac{2}{5}}$.

(d) $\mu_W[n] = (1-\alpha)\mu_X[n] = 0.5(1-\alpha)$. $R_W[m] = (-\alpha\delta[m+1] + (1+\alpha^2)\delta[m] - \alpha\delta[m-1])$ convloved with $R_X[m]$ which is equal to $.25(1 - 2\alpha + \alpha^2) + (1+\alpha^2)\exp(-|m|/5) - \alpha(\exp(-|m-1|/5) - \alpha(\exp(-|m+1|/5)$.

Problem 6.2 (8.2 in Stark and Woods)

The book's explanation below seems a bit of overkill. For a simpler answer, consider the random variables X, Y, and Z derived as follows: X is the number of heads on coin flip 1. Y is the number of heads on independent coin flip 2. Z is the sum of the number of heads on the two flips modulo 2. Clearly they are pairwise independent. But knowledge of any two determines the third.

Book answer:

Given the N−dimensional vector $(x_1, x_2, ..., x_N)$ whose components are *pairwise independent*,

$$\text{i.e.} \quad f(x_i, x_j) = f(x_i)f(x_j) \quad \text{for all } i \neq j,$$

we want to show that it is possible that,

$$f(x_1, x_2, ..., x_N) \neq f(x_1)f(x_2)\cdots f(x_N)$$

i.e. *joint independence* does not follow. Consider a case with $N = 3$: $f(x_3, x_2, x_1)$. By the chain rule for pdf's we then have $f(x_3, x_2, x_1) = f(x_3|x_2, x_1)f(x_2|x_1)f(x_1)$ and from pairwise independence we have $f(x_2, x_1) = f(x_2)f(x_1)$, $f(x_3, x_1) = f(x_3)f(x_1)$, and $f(x_3, x_2) = f(x_3)f(x_2)$, substituting in, we conclude

$$f(x_3, x_2, x_1) = f(x_3|x_2, x_1)f(x_2)f(x_1).$$

The question is now whether $f(x_3, x_1) = f(x_3)f(x_1)$, and $f(x_3, x_2) = f(x_3)f(x_2)$ provide enough information to conclude $f(x_3|x_2, x_1) = f(x_3)$. Alas, this is not so. [1]

Here is a specific counterexample: Let X_1 and X_2 be two independent RVs, each uniformly distributed on the interval $[-\pi, +\pi]$, i.e. $X_i : U[-\pi, +\pi], i = 1, 2$. In terms of pdf's, we have

$$f_{X_i}(x_i) = \begin{cases} \frac{1}{2\pi}, & |x_i| \leq \pi, \\ 0, & \text{else.} \end{cases}$$

Next, define a third RV by $X_3 \triangleq (X_1 + X_2) \bmod \pi$, meaning

$$X_3 = \begin{cases} X_1 + X_2 - 2\pi, & X_1 + X_2 > \pi, \\ X_1 + X_2, & |X_1 + X_2| \le \pi, \\ X_1 + X_2 + 2\pi, & X_1 + X_2 < -\pi. \end{cases}$$

Upon some reflection, we see

$$f_{X_3|X_1}(x_3|x_1) = \frac{1}{2\pi}, \quad |x_3| \le \pi,$$

and the same for $f_{X_3|X_2}$, and thus since X_1 and X_2 are independent, we can conclude that X_1, X_2, X_3 are pairwise independent. However, by the definition of X_3, we see that $(X_1 + X_2) \bmod \pi$ determines X_3, specifically

$$f_{X_3|X_1,X_2}(x_3|x_1, x_2) = \delta(x_3 - (x_1 + x_2) \bmod \pi).$$

Thus, joint independence does not prevail.

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[1] There is one exception to this and that is the case where the RVs are jointly Gaussian distributed.

Problem 6.3 (8.6 in Stark and Woods)

(a) Let events S_1 and S_2 be defined as follows for two times $t_2 > t_1 > 0$:

$$\begin{aligned} S_1 &\triangleq \{\text{no photon emitted prior to time } t_1\} \\ S_2 &\triangleq \{\text{ at least one photon emitted prior to time } t_2\}. \end{aligned}$$

By definition

$$\begin{aligned} P[S_2|S_1] &= \frac{P[S_2 S_1]}{P[S_1]} \text{ and} \\ P[S_1] &= 1 - \int_0^{t_1} \lambda e^{-\lambda t} dt = e^{-\lambda t_1}. \end{aligned}$$

Thus

$$P[S_2 S_1] = \int_{t_1}^{t_2} \lambda e^{-\lambda t} dt = e^{-\lambda t_1} - e^{-\lambda t_2}$$

and so

$$\begin{aligned} P[S_2|S_1] &= \frac{e^{-\lambda t_1} - e^{-\lambda t_2}}{e^{-\lambda t_1}} \\ &= 1 - e^{-\lambda(t_2 - t_1)}. \end{aligned}$$

(b) Let us define four events as follows:

$$\begin{aligned}
A &\triangleq \{\text{at least one photon emitted prior to time } t_2 \text{ from 3 independent sources}\},\\
S_1 &\triangleq \{\text{no photon emitted from source 1 prior to time } t_2\},\\
S_2 &\triangleq \{\text{no photon emitted from source 2 prior to time } t_2\}, \quad \text{and}\\
S_3 &\triangleq \{\text{no photon emitted from source 3 prior to time } t_2\}.
\end{aligned}$$

Then $P[A] = 1 - P[S_1S_2S_3]$, and because the three sources are independent $P[S_1S_2S_3] = P[S_1]P[S_2]P[S_3]$. Furthermore $P[S_i] = 1 - \int_0^{t_2} \lambda e^{-\lambda t} dt = e^{-\lambda t_2}$. Thus

$$\begin{aligned}
P[A] &= 1 - P[S_1]P[S_2]P[S_3]\\
&= 1 - e^{-3\lambda t_2}.
\end{aligned}$$

Problem 6.4 (8.14 in Stark and Woods)

(a) Denoting the outcomes as ζ_i, we have

$$\begin{aligned}
\mu_X[n] &\triangleq E[X[n]]\\
&= \sum_{\zeta_i} P[\{\zeta_i\}]X[n, \zeta_i]\\
&= \frac{1}{3}\left(3\delta[n] + u[n-1] + \cos\frac{\pi n}{2}\right).
\end{aligned}$$

(b)

$$\begin{aligned}
R_X[m,n] &\triangleq E[X[m]X^*[n]]\\
&= \sum_{\zeta_i} P[\{\zeta_i\}]X[m, \zeta_i]X^*[n, \zeta_i]\\
&= \frac{1}{3}\left(9\delta[m]\delta[n] + u[m-1]u[n-1] + \cos\frac{\pi m}{2}\cos\frac{\pi n}{2}\right).
\end{aligned}$$

(c) We can summarize the RVs $X[0]$ and $X[1]$ with the following table.

ζ_i	p	$X[0]$	$X[1]$
a	$\frac{1}{3}$	3	0
b	$\frac{1}{3}$	1	1
c	$\frac{1}{3}$	0	0

Thus $P[X[0] = 3, X[1] = 0] = P[\{a\}] = \frac{1}{3}$. The respective marginal probabilities are found as $P[X[0] = 3] = \frac{1}{3}$ and $P[X[1] = 0] = \frac{2}{3}$. Multiplying, we find

$$\begin{aligned}
P[X[0] &= 3, X[1] = 0] = \frac{1}{3}\\
&\neq \frac{1}{3}\frac{2}{3}\\
&= P[X[0] = 3]P[X[1] = 0],
\end{aligned}$$

therefore the RVs $X[0]$ and $X[1]$ are not independent.

Problem 6.5 (8.15 in Stark and Woods)

(a) The random variables $X[n]$ and $X[n-1]$ are jointly Gaussian distributed with zero means and covariance matrix

$$\mathbf{K} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix} \quad \text{with} \quad |\rho| < 1.$$

The determinant of this matrix is $\det \mathbf{K} = \sigma^4(1-\rho^2)$, and the inverse matrix is found as

$$\mathbf{K}^{-1} = \frac{1}{\sigma^4(1-\rho^2)} \begin{bmatrix} \sigma^2 & -\rho\sigma^2 \\ -\rho\sigma^2 & \sigma^2 \end{bmatrix}.$$

We can then write their joint pdf as

$$f_X(x_n, x_{n-1}) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2\sigma^2(1-\rho^2)}\left(x_n^2 - 2\rho x_n x_{n-1} + x_{n-1}^2\right)\right).$$

Also the marginal pdf for $X[n-1]$ is given directly as

$$f_X(x_{n-1}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x_{n-1}^2}{2\sigma^2}\right).$$

We can the write the conditional density

$$\begin{aligned} f_X(x_n|x_{n-1}) &= \frac{f_X(x_n, x_{n-1})}{f_X(x_{n-1})} \\ &= \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2\sigma^2(1-\rho^2)}(x_n - \rho x_{n-1})^2\right), \end{aligned}$$

after recognizing the perfect square $x_n^2 - 2\rho x_n x_{n-1} + \rho^2 x_{n-1}^2 \equiv (x_n - \rho x_{n-1})^2$. Recognizing that this conditional density is $N(\rho x_{n-1}, \sigma^2(1-\rho^2))$, we can immediately write its conditional mean

$$E[X[n]|X[n-1]] = \rho X[n-1].$$

(b) This predictor minimizes the mean square error over all functions $gX[n-1))$, i.e. it minimizes $E[(X[n] - g(X[n-1])^2]$ over all functions g. c.f. Example 4.3-4.

Problem 6.6 (8.17 in Stark and Woods)

We need the joint pdf $f_T(t_2, t_1; 10, 5)$. Now

$$\begin{aligned} T[10] &= \sum_{k=1}^{10} \tau[k] \\ &= T[5] + \sum_{k=6}^{10} \tau[k]. \end{aligned}$$

Calling $X \triangleq \sum_{k=6}^{10} \tau[k]$, we see that by definition, X and $T[5]$ are independent. This since $T[5]$ is a sum of earlier $\tau[k]'s$ not included in the sum that is X. Thus

$$\begin{aligned} f_T(t_2, t_1; 10, 5) &= f_T(t_2|t_1; 10, 5) f_T(t_1; 5) \\ &= f_T(t_2 - t_1; (10 - 5)) f_T(t_1; 5) \\ &= f_T(t_2 - t_1; 5) f_T(t_1; 5) \\ &= \frac{(\lambda (t_2 - t_1))^4}{4!} \lambda e^{-\lambda(t_2 - t_1)} \frac{(\lambda t_1)^4}{4!} \lambda e^{-\lambda t_1}, \quad t_2 \geq t_1 \geq 0. \end{aligned}$$