ECE 4260 Problem Set 5 Solutions

Problem 5.1 (5.1 in Stark and Woods)

The joint density function of n variables $x_i, i = 1, \ldots, n$ is given by

$$f_{\mathbf{X}}(\mathrm{x}) = Ke^{-\mathrm{x}^T \Lambda} u(\mathrm{x}) = Ke^{-\sum_{i=1}^{n} x_i \lambda_i} u(x_1) \ldots u(x_n).$$

Clearly, if K is a non-negative constant, the pdf also would be non-negative. In order that $f_{\mathbf{X}}(\mathrm{x})$ be a pdf, it should also integrate to 1. Therefore,

$$\int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathrm{x})\, d\mathrm{x} = 1 = K \int_{0}^{\infty} e^{-x_1 \lambda_1}\, dx_1 \ldots \int_{0}^{\infty} e^{-x_n \lambda_n}\, dx_n = \frac{K}{\prod_{i=1}^{n} \lambda_1}.$$

Hence only for $K = \prod_{i=1}^{n} \lambda_i$ is $f_{\mathbf{X}}(\mathrm{x})$ a valid pdf.

Problem 5.2 (5.3 in Stark and Woods)

We note that this multi-dimensional Gaussian is factorable so that

$$f_{\mathbf{X}}(\mathrm{x}) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{x_1^2}{2\sigma_1^2}} \ldots \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{x_n^2}{2\sigma_n^2}}.$$

With $g(x, \sigma) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$ (the Gaussian pdf with mean 0 and variance σ^2), we write

$$f_{\mathbf{X}}(\mathrm{x}) = g(x_1, \sigma_1) \ldots g(x_n, \sigma_n),$$

where $\int_{-\infty}^{\infty} g(x_i, \sigma_1)\, dx_i = 1$. Any marginal pdf $g(x_j, \sigma_j)$ can be obtained by

$$\begin{aligned}
f_{X_j}(x_j) &= \int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathrm{x})\, dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n \\
&= g(x_j, \sigma_j) \int_{-\infty}^{\infty} g(x_1, \sigma_1)\, dx_1 \ldots \int_{-\infty}^{\infty} g(x_{j-1}, \sigma_{j-1})\, dx_{j-1} \\
&\quad \times \int_{-\infty}^{\infty} g(x_{j+1}, \sigma_{j+1})\, dx_{j+1} \ldots \int_{-\infty}^{\infty} g(x_n, \sigma_n)\, dx_n \\
&= g(x_j, \sigma_j).
\end{aligned}$$

Problem 5.3 (5.6 in Stark and Woods)

From class. The joint density of max and min for n i.i.d. random variables can be inferred from the following reasoning:

One of the RVs is between z_1 and z_1+dz_1, one is between z_n and z_n+dz_n, and the rest are between z_1 and z_n. Therefore $f(z_1,z_n)dz_1dz_n=(n)(n-1)$[area between z_1 and z_n]$^{(n-2)}$ dz_1dz_n. In our case, the area is simply equal to z_n-z_1.

$f(z_1,z_n) = (n)(n-1)(z_n-z_1)^{(n-2)}$; $0 < z_1 < z_n < 1$.

We are given $f_{Z_1Z_2\cdots Z_n}(z_1, z_2, \cdots, z_n) = n! z_1 < z_2 < \cdots < z_n$ and 0 else. To get $f_{Z_1Z_n}(z_1, z_n)$ we integrate out with respect to $z_2, z_3, ..., z_{n-1}$. Thus

$$f_{Z_1Z_n}(z_1, z_n) = n! \int_{z_{n-1}}^{z_n} \left(\cdots \left(\int_{z_1}^{z_3} dz_2 \right) \cdots \right) dz_{n-1}.$$

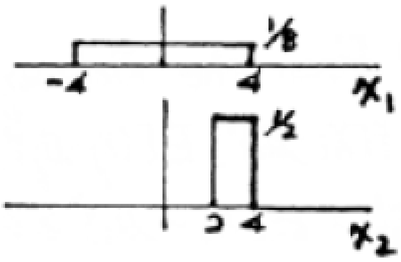
Let's take the first integration and leave out the $n!$:$\int_{z_1}^{z_3} dz_2 = z_3 - z_1$. The second integration yields $\int_{z_1}^{z_4} (z_3 - z_1) dz_3 = \frac{(z_4-z_1)^2}{2}$. The third integration yields $\int_{z_1}^{z_5} (z_4 - z_1)^2/2dz_3 = \frac{(z_5-z_1)^3}{3\cdot 2}$. Thus after $n-2$ integrations we end up with $\int_{z_1}^{z_n} \frac{(z_{n-1}-z_1)^{n-3}}{n-3} dz_{n-1} = \frac{(z_n-z_1)^{n-2}}{(n-2)\cdots 3\cdot 2}$. Putting back the $n!$ then yields

$$f_{Z_1Z_n}(z_1, z_n) = \begin{cases} n(n-1)(z_n - z_1)^{n-2}, & 0 < z_1 < z_n < 1, n \geq 2 \\ 0, & \text{else.} \end{cases}$$

Problem 5.4 (5.19 in Stark and Woods)

We can write:

$$\begin{aligned} f_{X_1}(x_1) &= \frac{1}{8}[u(x_1 + 4) - u(x_1 - 4)] \\ f_{X_2}(x_2) &= \frac{1}{2}[u(x_2 - 2) - u(x_2 - 4)] \end{aligned}$$



Then $f_{X_1}(x_1)f_{X_2}(x_2) = f_{X_1,X_2}(x_1, x_2)$ and the RVs X_1 and X_2 are independent. Hence, they are uncorrelated,

$$\begin{aligned} E[X_1X_2] &= E[X_1]E[X_2] \\ &= 0 \end{aligned}$$

since $E[X_1] = 0$. Thus, X_1 and X_2 are orthogonal.

Another way to approach it is simply to look at the symmetry. The pdf factors and the non-zero region is rectangular. Hence they are statistically (and linearly) independent. Linear independence along with at least one mean of zero implies orthogonality.

Problem 5.5 (5.23 in Stark and Woods)

(a) From the Schwarz inequality,

$$\begin{aligned} \sigma_i^2\sigma_j^2 &= E[(X_i-\mu_i)^2]E[(X_j-\mu_j)^2] \\ &\geq \left|E[(X_i-\mu_i)(X_j-\mu_j)]\right|^2 \\ &= |K_{ij}|^2. \end{aligned}$$

But in the given matrix, $\sigma_1^2\sigma_2^2 = 2\times 3 = 6$ and $|K_{12}|^2 = 16 > \sigma_1^2\sigma_2^2$. Thus violating the Schwarz inequality.

(b) Always $\sigma_{33}^2 = E[(X_3-\mu_3)^2] \geq 0$. In the given matrix, this value is -2.

(c) The covariance value $K_{12} = E[(X_1-\mu_1)(X_2-\mu_2)]$ must be real for a real valued random vector. In the given matrix, there are numbers with non-zero imaginary parts.

(d) Always for covariance matrices of real valued random vectors, we have the symmetry conditions

$$\begin{aligned} K_{ij} &= E[(X_i-\mu_i)(X_j-\mu_j)] \\ &= E[(X_j-\mu_j)(X_i-\mu_i)] \\ &= K_{ji}. \end{aligned}$$

But, in the given matrix, $K_{23} = 3, K_{32} = 12 \neq K_{23}$, thus violating the required symmetry.

Problem 5.6 (5.29 in Stark and Woods)

The mean of $\mathbf{Y}$ is given by

$$E[\mathbf{Y}] = E[\mathbf{A}^T\mathbf{X}+B] = \mathbf{A}^T\boldsymbol{\mu}+B = \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ -5 \\ 6 \end{pmatrix} + 5 = 32.$$

Let $E[\mathbf{Y}] = \boldsymbol{\mu}_1$. Then

$$\begin{aligned} \sigma_Y^2 &= E\left[(\mathbf{Y}-\boldsymbol{\mu}_1)^T(\mathbf{Y}-\boldsymbol{\mu}_1)\right] \\ &= E\left[\left(\mathbf{A}^T(\mathbf{X}-\boldsymbol{\mu})\right)^T\left(\mathbf{A}^T(\mathbf{X}-\boldsymbol{\mu})\right)\right] \\ &= E\left[\left((\mathbf{X}-\boldsymbol{\mu})^T\mathbf{A}\right)\left(\mathbf{A}^T(\mathbf{X}-\boldsymbol{\mu})\right)\right] \\ &= E\left[\left(\mathbf{A}^T(\mathbf{X}-\boldsymbol{\mu})\right)\left((\mathbf{X}-\boldsymbol{\mu})^T\mathbf{A}\right)\right] \\ &= \mathbf{A}^T E\left[(\mathbf{X}-\boldsymbol{\mu})^T(\mathbf{X}-\boldsymbol{\mu})\right]\mathbf{A} \\ &= \mathbf{A}^T\mathbf{K}_\mathbf{X}\mathbf{A} \\ &= \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 25. \end{aligned}$$

Problem 5.7 (5.24 in Stark and Woods part a)

(a)

$$\begin{aligned}\det(\mathbf{K_{XX}} - \lambda\mathbf{I}) &= \det\begin{bmatrix} 3-\lambda & \sqrt{2} \\ \sqrt{2} & 4-\lambda \end{bmatrix} = 0 \\ &\Rightarrow \quad \lambda^2 - 7\lambda + 10 = 0, \text{ or } \lambda_1 = 5, \text{ and } \lambda_2 = 2.\end{aligned}$$

i) $\lambda_1 = 5$: $\quad \mathbf{K_{XX}}\boldsymbol{\phi}_1 = 5\boldsymbol{\phi}_1$ or what is the same, $(\mathbf{K_{XX}} - 5\mathbf{I}) = \mathbf{0}$ leads to

$$\begin{aligned}-2\phi_{11} + \sqrt{2}\phi_{12} = &\quad 0 \\ \sqrt{2}\phi_{11} - \phi_{12} = &\quad 0 \\ \Rightarrow \quad \boldsymbol{\phi}_1 &= (\phi_{11}, \phi_{12})^T \\ &= \frac{1}{\sqrt{3}}(1, \sqrt{2})^T \quad .\end{aligned}$$

ii) $\lambda_2 = 2$: $\quad \mathbf{K_{XX}}\boldsymbol{\phi}_1 = 2\boldsymbol{\phi}_1$ or what is the same, $(\mathbf{K_{XX}} - 2\mathbf{I}) = \mathbf{0}$ leads to

$$\begin{aligned}\phi_{21} + \sqrt{2}\phi_{22} = &\quad 0 \\ \sqrt{2}\phi_{21} + 2\phi_{22} = &\quad 0 \\ \Rightarrow \quad \boldsymbol{\phi}_2 &= (\phi_{21}, \phi_{22})^T \\ &= \sqrt{\frac{2}{3}}(1, -\frac{1}{\sqrt{2}})^T \quad .\end{aligned}$$

Thus

$$\begin{aligned}\boldsymbol{\Phi} &= \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}.\end{aligned}$$

Set

$$\begin{aligned}\boldsymbol{\Lambda}^{-\frac{1}{2}} &\triangleq \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix},\end{aligned}$$

and the define

$$\mathbf{C} \triangleq \boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Phi}^T.$$

Of course, the `eig` command in Matlab makes this much simpler.

Then with $\mathbf{Y} = \mathbf{C}\mathbf{X}$, we have

$$\begin{aligned}
E[\mathbf{Y}\mathbf{Y}^T] &= \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{\Phi}^T E[\mathbf{X}\mathbf{X}^T]\mathbf{\Phi}\mathbf{\Lambda}^{-\frac{1}{2}} \\
&= \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{\Phi}^T \mathbf{K}_{\mathbf{XX}}\mathbf{\Phi}\mathbf{\Lambda}^{-\frac{1}{2}} \\
&= \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{\Phi}^T (\mathbf{\Phi}\mathbf{\Lambda})\,\mathbf{\Lambda}^{-\frac{1}{2}} \\
&= \mathbf{\Lambda}^{-\frac{1}{2}}(\mathbf{\Phi}^T\mathbf{\Phi})\mathbf{\Lambda}\mathbf{\Lambda}^{-\frac{1}{2}} \\
&= \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{I}\mathbf{\Lambda}\mathbf{\Lambda}^{-\frac{1}{2}} \\
&= \mathbf{I}.
\end{aligned}$$

Thus $\mathbf{Y} = \mathbf{C}\mathbf{X} \;\Rightarrow\; \mathbf{K}_{\mathbf{XX}} = \mathbf{I}$. Finally, since $\mathbf{C}^T \neq \mathbf{C}^{-1}$, $\mathbf{C}$ is not unitary.

Problem 5.8 (5.31 in Stark and Woods)

The mean of Y is given as

$$E[Y] = E[\sum_{i=1}^{n} p_i X_i] = \sum_{i=1}^{n} p_i E[X_i] = \sum_{i=1}^{n} p_i \mu_i.$$

The variance of Y is given by

$$\begin{aligned}
\sigma_Y^2 &= E[(Y - \mu_Y)^2] \\
&= E\left[\sum_{i=1}^{n} p_i (X_i - \mu_i)\right]^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j E[(X_i - \mu_i)(X_j - \mu_j)] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j K_{ij}.
\end{aligned}$$