ECE 4260 Problem Set 4 Solutions

Problem 4.1 (4.21 a, b, and c in Stark and Woods)

(a) $E[X^2] = 1^2 p + 0^2(1-p) = p$, $E[X] = p$, so $\sigma^2 = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$.

(b) For the binomial random variable X, we have $E[X] = np$ from Problem 4.4. Then, letting $q = 1-p$, we have

$$\begin{aligned}
E[X^2] &= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} p^k q^{n-k} \\
&= \sum_{k=1}^{n} k \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k} \\
&= \sum_{k=1}^{n} ((k-1)+1) \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k} \\
&= n(n-1)p^2 + np.
\end{aligned}$$

Then $\sigma^2 = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - (np)^2 = np(1-p) = npq$.

(c) From Example 4.1-6, we have $E[X] = a$ for the Poisson random variable with parameter $a(> 0)$. Remember the Poisson PMF is given as $P_X(k) = \frac{a^k}{k!}e^{-a}u(k)$. To compute the second moment, we proceed

$$\begin{aligned}
E[X^2] &= \sum_{k=-\infty}^{+\infty} k^2 P_X(k) \\
&= \sum_{k=1}^{\infty} k^2 \frac{a^k}{k!} e^{-a} \quad \text{since the } k = 0 \text{ term will be zero,} \\
&= a\left(\sum_{k=1}^{\infty} (k-1+1)\frac{a^{k-1}}{(k-1)!}\right) e^{-a} \\
&= ae^{-a}\left(\sum_{k=1}^{\infty} (k-1)\frac{a^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} 1\frac{a^{k-1}}{(k-1)!}\right) \\
&= ae^{-a}\left(\sum_{k'=0}^{\infty} k'\frac{a^{k'}}{k'!} + \sum_{k'=0}^{\infty} 1\frac{a^{k'}}{k'!}\right) \quad \text{with the substitution } k' = k-1, \\
&= ae^{-a}(ae^{+a} + e^{+a}) \\
&= a^2 + a.
\end{aligned}$$

Then $\sigma^2 = E[X^2] - (E[X])^2 = a^2 + a - a^2 = a(> 0)$.

Problem 4.2 (4.27 in Stark and Woods)

(a) $\mu_Z = E[Z] = E[\frac{1}{2}(X+Y)] = \frac{1}{2}E[X] + \frac{1}{2}E[Y] = 0$. The variance is given by

$$\begin{aligned}
\sigma_Z^2 &= E[Z^2] - \mu_Z^2 \\
&= E\left[\left(\frac{X+Y}{2}\right)^2\right] - 0 \\
&= \frac{1}{4}E[X^2 + 2XY + Y^2] \\
&= \frac{1}{4}(\sigma^2 + 2E[XY] + \sigma^2) \\
&= \frac{1}{4}(2\sigma^2 + 0) \\
&\quad \text{(since } X, Y \text{ are independent,} E[XY] = E[X]E[Y]) \\
&= \frac{\sigma^2}{2}.
\end{aligned}$$

(b) Even if X and Y are dependent, the mean of Z would remain the same. Hence, $E[Z] = 0$. The variance of Z is given by

$$
\begin{aligned}
\sigma_Z^2 &= E[\frac{1}{4}(X+Y)^2] - 0 \\
&= \frac{1}{4}E[X^2 + 2XY + Y^2] \\
&= \frac{1}{4}(\sigma^2 + 2E[XY] + \sigma^2) \\
&= \frac{1}{4}(2\sigma^2 + 2\rho\sigma^2) \\
&\quad (\text{because } E[(X-0)(Y-0)] = \rho\sigma_X\sigma_Y = \rho\sigma^2) \\
&= \frac{1}{2}\sigma^2(1+\rho).
\end{aligned}
$$

(c) For $\rho = -1$, 0, and 1, the values of σ_Z^2 are 0, $\frac{\sigma^2}{2}$, and σ^2, respectively. The variance of the sample average does not reduce when the random variables are perfectly correlated. If X and Y are uncorrelated, $\rho = 0$, and since they are Gaussian, they are independent. In that case, the variance of the sample average goes down with the number of samples. For values of ρ other than 1, the variance of the sample average is less than the variance of each sample.

Problem 4.3 (4.30 in Stark and Woods)

The conditional mean is always the mean of the conditional density. Since this conditional density is $N(\alpha x, \sigma^2)$, it follows that the conditional mean is αx, i.e. $E[Y|X=x] = \alpha x$, then by definition of the conditional mean as a random variable, we have

$$E[Y|X] = \alpha X.$$

Problem 4.4 (4.34 in Stark and Woods)

From the text: $\alpha_o = \rho\frac{\sigma_Y}{\sigma_X}$ and $\beta_o = \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}\mu_X$. Then

$$
\begin{aligned}
\epsilon_{\min}^2 &= E[(Y - \alpha_o X - \beta_o)^2] \\
&= E[((Y-\mu_Y) - \alpha_o(X-\mu_X))^2] \\
&= \sigma_Y^2 - 2\rho\frac{\sigma_Y}{\sigma_X}\rho\sigma_X\sigma_Y + \left(\rho\frac{\sigma_Y}{\sigma_X}\right)^2\sigma_X^2 \\
&= \sigma_Y^2(1-\rho^2).
\end{aligned}
$$

When $\rho = 1$, then Y is a linear function of X and there is no error in prediction.

Problem 4.5 (4.46 in Stark and Woods)

Since $f_X(x) = \frac{1}{\pi(1+(x-a)^2)}$ is a pdf for any value of a including $a = 0$, *we* immediately deduce that $\int_{-\infty}^{\infty} \frac{1}{\pi(1+(x-a)^2)} dx = 1$. Then

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} \frac{x}{\pi(1+(x-a)^2)} dx \\
&= \int_{-\infty}^{\infty} \frac{u+a}{\pi(1+u^2)} du \\
&= \int_{-\infty}^{\infty} \frac{u}{\pi(1+u^2)} du + \int_{-\infty}^{\infty} \frac{a}{\pi(1+u^2)} du \\
&= 0 + a = a.
\end{aligned}
$$

We took advantage that the first integral after the third equal sign is 0 because the integrand is odd and the interval about zero is even. To calculate the variance we would have to consider an integral of the form $E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\pi(1+(x-a)^2)} dx$, which is readily seen not to converge in any sense. See what happens to the integrand when x approaches infinity.

Note, however, that the symmetry argument used yields only the principal value of the integral. An easier way to deduce the mean is by noticing that the pdf is symmetric around *a*, which should imply a mean of *a*. Again, this is only a principal value. Cauchy distributions and their ilk are notoriously difficult to analyze by applying standard techniques. Unfortunately, they often arise in nature.

Problem 4.6 (4.47 in Stark and Woods)

The solution proceeds as

$$
\begin{aligned}
\Phi_X(\omega) &\triangleq E[e^{+j\omega X}] \\
&= \int_0^{\infty} \frac{1}{\mu} e^{(j\omega - 1/\mu)x} dx \\
&= \frac{1}{\mu} \left. \frac{e^{(j\omega-1/\mu)x}}{j\omega - \frac{1}{\mu}} \right|_0^{\infty} \\
&= \frac{1}{\mu} \frac{-1}{j\omega - \frac{1}{\mu}} \\
&= \frac{1}{1 - j\omega\mu}, \qquad -\infty < \omega < +\infty.
\end{aligned}
$$

This is just the Fourier transform of a decaying exponential with a negative sign for ω.

To find the second moment, take the second derivative with respect to ϖ, multiply by -1, and set ϖ equal to 0. Here the second derivative is

$$\frac{2(j\mu)^2}{(1-j\omega\mu)^3}$$

Setting ϖ to 0 yields

$$2(j\mu)^2$$

Which when multiplied by -1, gives

$$2\mu^2$$

Problem 4.7 (4.54 in Stark and Woods)

Using only the properties of Poisson processes, we interpret the two RVs as follows: X represents the number of events for a Poisson process of rate 2 over an interval of length 1; Y represents the number of events for a Poisson process of rate 3 over an interval of length 1. Since they are independent, their sum would represent the number of events for a Poisson process of rate 5 over an interval of length 1, giving $e^{-5}5^k/k!$, k=0,1,2…

If we use MGFs or CFs, the result is shown below.

Two random variables X: Poisson(a) and Y: Poisson (b) are given, where $a = 2$, and $b = 3$, $Z = X + Y$, and the characteristic function of Z is given by

$$\begin{aligned}
\Phi_Z(\omega) &= E[e^{j\omega Z}] \\
&= E[e^{j\omega(X+Y)}] \\
&= E[e^{j\omega X}e^{j\omega Y}] \\
&= E[e^{j\omega X}]E[e^{j\omega Y}] \text{ (because } X \text{ and } Y \text{ are independent)} \\
&= \Phi_X(\omega)\Phi_Y(\omega).
\end{aligned}$$

The characteristic function of X is given by

$$\begin{aligned}
\Phi_X(\omega) &= E[e^{j\omega X}] \\
&= \sum_{k=0}^{\infty} e^{j\omega k} P_X(k) \\
&= \sum_{k=0}^{\infty} e^{j\omega k} \frac{e^{-a}a^k}{k!} \\
&= e^{-a} \sum_{k=0}^{\infty} \frac{(ae^{j\omega})^k}{k!} \\
&= e^{-a}e^{ae^{j\omega}} \\
&= \exp(-a + ae^{j\omega}) \\
&= \exp(-a(1 - e^{j\omega})).
\end{aligned}$$

Similarly, we obtain $\Phi_Y(\omega) = \exp(-b(1 - e^{j\omega}))$. Therefore,

$$\begin{aligned}
\Phi_Z(\omega) &= \exp(-a(1 - e^{j\omega}))\exp(-b(1 - e^{j\omega})) \\
&= \exp(-(a + b)(1 - e^{j\omega})).
\end{aligned}$$

This implies that Z is Poisson$(a + b)$ and has a density function $f_Z(z) = \frac{e^{-5}5^n}{n!}$, for $n = 0, 1, 2, \ldots$, for $a = 2$ and $b = 3$.

Problem 4.48 (4.55 in Stark and Woods)

For a single Bernoulli trial with p=0.05, the mean is 0.05 and the variance is (0.05)(0.95)=0.0475. The sum of 2000 of these would give a mean of 100, a variance of 95, and a standard deviation of 9.75. The probability that it is more than 109 is best approximated by use of the Demoivre-Laplace approximation of greater than 109.5. 109.5 is 9.5/9.75 = 0.974 standard deviations from the mean. We need 1-Φ(0.974) = 1-.835 = 0.165.