ECE 4260 Problem Set 3 Solutions

Problem 3.1: (3.24 in Stark and Woods)

(a) We have to calculate the running integral of the density $f_Y(y) = \frac{c}{2}\exp(-c|y|)$. Now

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} f_Y(u)du \\ &= \int_{-\infty}^{y} \frac{c}{2}\exp(-c|u|)du. \end{aligned}$$

Because of the absolute value sign, it is easier to consider the two cases $y \leq 0$ and $y \geq 0$ separately. First we evaluate for $y \leq 0$, where $f_Y(y) = \frac{c}{2}\exp(+cy)$. We find

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} \frac{c}{2}\exp(cu)du \\ &= \frac{c}{2}\left(\frac{1}{c}\exp(cu)|_{-\infty}^{y}\right) \\ &= \frac{1}{2}\exp(cy), \qquad \text{for } y \leq 0. \end{aligned}$$

Now we consider the case $y \geq 0$, where $f_Y(y) = \frac{c}{2}\exp(-cy)$. We note that by symmetry $\int_{-\infty}^{0} \frac{c}{2}\exp(cu)du = 1/2$, so we can write

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} \frac{c}{2}\exp(-c|u|)du \\ &= \frac{1}{2} + \int_{0}^{y} \frac{c}{2}\exp(-cu)du \\ &= \frac{1}{2} + \frac{c}{2}\left(\frac{1}{c}\exp(-cu)|_{0}^{y}\right) \\ &= \frac{1}{2} + \frac{1}{2}(1 - \exp(-cy)) \\ &= 1 - \frac{1}{2}\exp(-cy), \qquad \text{for } \; y \geq 0. \end{aligned}$$

Now, as a check, we note that both results agree at their common point $y = 0$ as they should. Overall, we can write the Laplacian distribution function as

$$F_Y(y) = \begin{cases} \frac{1}{2}\exp(cy), & y < 0, \\ 1 - \frac{1}{2}\exp(-cy), & y \geq 0. \end{cases}$$

(b) Probably the first thing to do here is to note that since $X : U[0,1]$, we have that $F_X(x) = x\{u(x) - u(x-1)\}$, i.e. just a straight line segment with slope 1 on $[0,1]$. Since the distribution function F_Y is monotone increasing, we have

$$F_Z(z) = P[X \leq g^{-1}(z)]$$

Next, we note that since $g = F_Y^{-1}$, so $g^{-1} = F_Y$ and hence

$$\begin{aligned} F_Z(z) &= P[X \leq g^{-1}(z)] \\ &= F_X(F_Y(z)) \\ &= F_Y(z)\,(u(F_Y(z)) - u(F_Y(z) - 1)) \\ &= F_Y(z)\,(1 - 0) \\ &= F_Y(z). \end{aligned}$$

(c) This method strictly speaking will not work with either jumps or flat regions in the desired distribution function F_Y. In a flat region of F_Y, the corresponding g would be a vertical line, not be a valid function! At a jump of F_Y, $g = F_Y^{-1}$ will not be defined for some of the input values. This won't work either. On the other hand, there are simple modifications of this method that can get around these problems and make the basic method useful in both cases. One simply has to remove the flat regions from F_Y before finding the inverse function. At the jumps, where the inverse function would have a gap, just fill it in with a horizontal line. With these changes the basic method extends to both mixed and discrete distribution functions.

Problem 3.2: (3.30 in Stark and Woods)

$Z_n \triangleq \max(X_1, X_2, ..., X_n)$ and the X_is are independent RVs. Then

$$\begin{aligned} F_{Zn}(z) &= P[Z_n \le z] \\ &= P[X_1 \le z]P[X_2 \le z] \cdots P[X_n \le z] \\ &= (F_X(z))^n \\ &= (1 - e^{-z})^n u(z). \end{aligned}$$

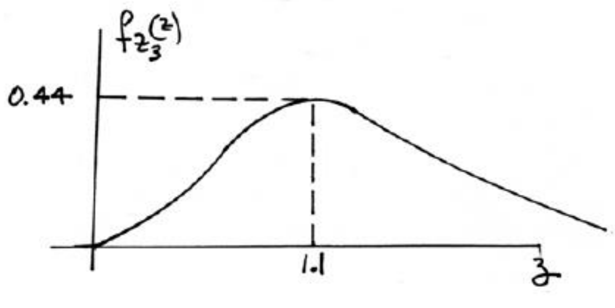
Hence

$$\begin{aligned} f_{Z_n}(z) &= \frac{dF_{Zn}(z)}{dz} \\ &= n((1 - e^{-z})^{n-1} e^{-z} u(z). \end{aligned}$$

The peak of this curve will occur at

$$\begin{aligned} 0 &= f'_{Z_n}(z) = \\ &= n(n-1)(1 - e^{-z})^{n-2} e^{-2z} - n(1 - e^{-z})^{n-1} e^{-z} \\ &= (n-1)e^{-z} - (1 - e^{-z}), \end{aligned}$$

which happens at $ne^{-z_o} = 1$ or $z_o = \ln(n)$. For $n = 3$, $z_o \simeq 1.1$ and $f_{Z_3}(z_o) \simeq 0.444$. See sketch below for $n = 3$.



As for the minimum, we have then $Z_n \triangleq \max(X_1, X_2, ..., X_n)$, where again the X_is are independent RVs. Then

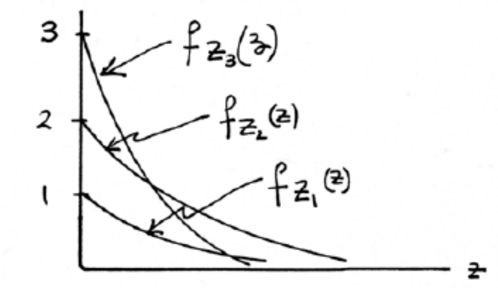
$$\begin{aligned} F_{Z_n}(z) &= P[Z_n \le z] \\ &= 1 - P[Z_n > z] \\ &= 1 - P[X_1 > z]P[X_2 > z] \cdots P[X_n > z] \\ &= 1 - (1 - F_{X_1}(z))(1 - F_{X_2}(z)) \cdots (1 - F_{X_n}(z)) \\ &= 1 - [1 - (1 - e^{-z})u(z)]^n \end{aligned}$$

$$= \begin{cases} 0, & z < 0, \\ (1 - e^{-nz}) & z \geq 0 \end{cases}$$
$$= (1 - e^{-nz})u(z).$$

So the pdf is given as

$$f_{Z_n}(z) = F'_{Z_n}(z)$$
$$= ne^{nz}u(z).$$

Here is a sketch for $n = 1, 2, 3$.



Additional Part:

Let $U = \min(X_1, X_2, ...X_n)$; $V = \max(X_1, X_2, ...X_n)$;

$$f_{UV}(u, v) = n(n-1)f_X(u)f_X(v)[\int_u^v f_X(x)dx]^{n-2}$$
$$0 \leq u \leq v$$
$$= n(n-1)(e^{-u})(e^{-v})[e^{-v} - e^{-u}]^{n-2}$$
$$0 \leq u \leq v$$

Problem 3.3 (3.36 in Stark and Woods)

(a)

We look at the transformation problem for two independent Normal random variables X and $Y : N(0, \sigma^2)$, transformed to $Z \triangleq X^2 + Y^2$ and $W \triangleq X$. We thus have

$$z = g(x, y) = x^2 + y^2 \quad \text{and} \quad w = h(x, y) = x.$$

This is a non-invertible transformation with two real roots, for $|w| < \sqrt{z}, z > 0$,

$$
\begin{aligned}
R_1 &: \quad x = w, y = +\sqrt{z - w^2}, \text{ and} \\
R_2 &: \quad x = w, y = -\sqrt{z - w^2}.
\end{aligned}
$$

Now at both roots the magnitude of the Jacobian is the same,

$$
\begin{aligned}
|J_1| &= |J_2| = 2\sqrt{z - w^2}, \text{ where} \\
J_{1,2} &= \det \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} = 2y = \pm 2\sqrt{z - w^2}.
\end{aligned}
$$

(b)

Hence

$$
\begin{aligned}
f_{Z,W}(z, w) &= \frac{1}{2\sqrt{z - w^2}} \left(f_{X,Y}(w, \sqrt{z - w^2}) + f_{X,Y}(w, -\sqrt{z - w^2}) \right) \\
&= \begin{cases} \frac{1}{2\pi\sigma^2} \frac{1}{\sqrt{z - w^2}} \exp(-z/2\sigma^2), & |w| < \sqrt{z}, z > 0 \\ 0, & \text{else.} \end{cases}
\end{aligned}
$$

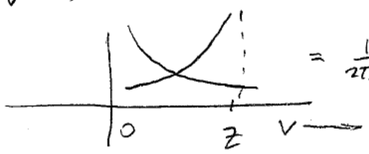
We can find the marginal density f_Z either by integrating out the unwanted variable in this joint density, or by using the result of Example 3.3-10 that Z will be Exponential distributed (equivalently Chi-square with 2 degrees of freedom). Either way the answer is

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{+\infty} f_{Z,W}(z, w) dw \\
&= \frac{1}{2\pi\sigma^2} \int_{-\sqrt{z}}^{+\sqrt{z}} \frac{1}{\sqrt{z - w^2}} \exp(-z/2\sigma^2) dw \\
&= \frac{1}{2\sigma^2} \exp(-z/2\sigma^2)\, u(z).
\end{aligned}
$$

(b) For $X \sim N(0,\sigma^2)$, and $V = X^2$,

$$f_V(v) = \frac{v^{-1/2}}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}v/\sigma^2} u(v)$$

$$f_Z(z) = f_V(z) * f_V(z) = \frac{1}{2\pi\sigma^2}\int_0^z v^{-1/2} e^{-\frac{1}{2}v/\sigma^2} (z-v)^{-1/2} e^{-\frac{1}{2}\frac{(z-v)}{\sigma^2}} dv$$



$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{z}{\sigma^2}} \int_0^z (v(z-v))^{-1/2} dv \qquad z \geq 0$$

The answer according to Example 3.3-10 is:

$$f_Z(z) = \frac{1}{2\sigma^2} e^{-z/2\sigma^2} u(z)$$

$\therefore \int_0^z (v(z-v))^{-1/2} dv$ must equal π.

Problem 3.4 (3.37 in Stark and Woods) (see notes at end)

Book's solution below.

We want to find the pdf of the two variables

$$Z = aX + bY$$

$$W = cX + dY$$

The joint pdf of X and Y are given as

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} e^{-Q(x,y)},$$

where $Q(x,y) = \frac{1}{2\sigma^2(1-\rho^2)}[x^2 - 2\rho xy + y^2]$. Consider the inverse transformation, i.e.,

$$X = \hat{a}Z + \hat{b}W$$

$$Y = \hat{c}Z + \hat{d}W$$

Note that from the above transformation, the solution of the two equations are given as $x = \hat{a}z + \hat{b}w$ and $y = \hat{c}z + \hat{d}w$. The term in the exponent of the pdf will be $\frac{1}{2\sigma^2(1-\rho^2)}[x^2 - 2\rho xy + y^2]$, and this is given as

$$\frac{1}{2\sigma^2(1-\rho^2)}[x^2 - 2\rho xy + y^2] = \frac{1}{2\sigma^2(1-\rho^2)}\left[(\hat{a}z + \hat{b}w)^2 - 2\rho(\hat{a}z + \hat{b}w)(\hat{c}z + \hat{d}w) + (\hat{c}z + \hat{d}w)^2\right].$$

In this exponent, if the cross terms (terms that contain zw) vanish, then we would be able to split the pdf in to the product of the two marginal pdf's. In other words, if the coefficients of the terms zw is zero, then we would be able to write $f_{Z,W} = f_Z f_W$. Therefore, we need

$$(2\hat{a}\hat{b} + 2\hat{c}\hat{d} - 2\rho\hat{a}\hat{d} - 2\rho\hat{b}\hat{c}) = 0$$

*****The solution below has a typo. It should read a = -b, c = d, or a = b, c = - d. *****

. If we chose $\hat{a} = \hat{c}, \hat{b} = -\hat{d}$, the coefficient of zw will be zero. Then

$$x = \hat{a}z + \hat{b}w$$

$$y = \hat{a}z - \hat{b}w$$

will give us $\frac{1}{2\sigma^2(1-\rho^2)}[x^2 + y^2 - 2\rho xy] = \frac{1}{2\sigma^2(1-\rho^2)}[2(1-\rho)\hat{a}^2 z^2 + 2(1+\rho)\hat{b}^2 w^2]$. Therefore, $Q(x, y) = \frac{1}{2}\left\{\left[\frac{z\sqrt{1-\rho}}{\sigma/\sqrt{2}\hat{a}}\right]^2 + \left[\frac{w\sqrt{1+\rho}}{\sigma/\sqrt{2}\hat{b}}\right]^2\right\}$. The magnitude of the Jacobian is given as

$$|J| = \text{mag}\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \frac{1}{2|\hat{a}\hat{b}|},$$

where $g(x, y) = \frac{x+y}{2\hat{a}}$, $h(x, y) = \frac{x-y}{2\hat{b}}$. With $\sigma_1 \triangleq \sigma\sqrt{1-\rho}/\sqrt{2}\hat{a}$, $\sigma_2 \triangleq \sigma\sqrt{1+\rho}/\sqrt{2}\hat{b}$. Hence,

$$\begin{aligned} f_{ZW}(z, w) &= \frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{1}{2}\frac{z^2}{\sigma_1^2}}\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{1}{2}\frac{w^2}{\sigma_2^2}} \\ &= f_Z(z)f_W(w). \end{aligned}$$

Solutions as required in problem statement:

(a) with the description given, we need

$E(ZW) = 0 = E\{(aX+bY)(cX+dY)\}$

$= E\{acX^2 + bdY^2 + (ad+bc)XY\}$

$= ac\sigma^2 + bd\sigma^2 + (ad+bc)\rho\sigma^2$

$= (ac+bd)\sigma^2 + (ad+bc)\rho\sigma^2 = 0$

Easy solution: make $a=1$, $b=1$, $c=1$, $d=-1$

(b) Since $\sigma_Y = \sigma_X$, the ellipse will point at $\pm 45°$, independent of ρ. $\therefore$ a $45°$ rotation is needed. $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ does such a rotation, as well as any scale. For there to be no expansion, use: $\begin{pmatrix} \frac{\sqrt{2}}{2} & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$

These are the eigenvectors of $K = \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}$

Problem 3.5 (3.38 in Stark and Woods)

Let $g(x,y) \triangleq \frac{x^2+y^2}{2}$ and $h(x,y) \triangleq \frac{x^2-y^2}{2}$. The real roots of $g(x,y) = v$, $h(x,y) = w$ occur for $v \geq 0, |v| \geq |w|$ and are four in number.

$$\begin{aligned} x_1 &= +\sqrt{v+w}, \quad y_1 = +\sqrt{v-w} \\ x_2 &= -\sqrt{v+w}, \quad y_2 = +\sqrt{v-w} \\ x_3 &= -\sqrt{v+w}, \quad y_3 = -\sqrt{v-w} \\ x_4 &= +\sqrt{v+w}, \quad y_4 = -\sqrt{v-w}. \end{aligned}$$

We note that w can be negative, but never greater in magnitude than v. The magnitude of J is

$$\text{abs}\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = 2|xy| = 2\sqrt{v^2 - w^2},$$

and this is the same for all the four roots. Now observe: $x^2 + y^2 = 2v$ for all the roots, and

$$x_1 y_1 = \sqrt{v^2 - w^2} = x_3 y_3$$

$$x_2 y_2 = -\sqrt{v^2 - w^2} = x_4 y_4$$

Hence

$$\begin{aligned} f_{VW}(v,w) &= \frac{1}{4\pi\sqrt{v^2-w^2}\sqrt{1-\rho^2}} 2\left\{ e^{-\left[\frac{2v-2\sqrt{v^2-w^2}\rho}{2(1-\rho^2)}\right]} + e^{-\left[\frac{2v+2\sqrt{v^2-w^2}\rho}{2(1-\rho^2)}\right]} \right\} \\ &= \frac{1}{\pi\sqrt{(1-\rho^2)(v^2-w^2)}} e^{-\frac{v}{1-\rho^2}} \cosh\left(\frac{\rho\sqrt{v^2-w^2}}{1-\rho^2}\right), \quad \text{for } v \geq 0, |v| \geq |w|, \end{aligned}$$

where the hyperbolic cosine function $\cosh(x) \triangleq \frac{1}{2}(e^x + e^{-x})$. For $v < 0$ or $|v| < |w|$, there are no real roots of the transformation equations, so that we have $f_{VW}(v,w) = 0$ there.

Clearly they are not independent.

Problem 3.6 (4.19 in Stark and Woods)

Let the number of units manufactured at the various sites be denoted n_A, n_B, and n_C, with total number of units simply n. Then from the problem statement we know that

$$n_A = 3n_B \quad \text{and} \quad n_B = 2n_C,$$

and of course $n = n_A + n_B + n_C$. Then from classical probabilities, we get the probability of a unit selected 'at random' as

$$P[A] = \frac{n_A}{n} = \frac{6}{9}, P[B] = \frac{n_B}{n} = \frac{2}{9}, \text{ and } P[C] = \frac{n_C}{n} = \frac{1}{9},$$

where we define event $A \triangleq \{\text{unit comes from plant } A\}$, and so forth for events B and C. Now we can use the concept of conditional expectation to write

$$\begin{aligned} E[X] &= E[X|A]P[A] + E[X|B]P[B] + E[X|C]P[C] \\ &= \frac{1}{5}\int_0^\infty xe^{-x/5}dx\frac{6}{9} + \frac{1}{6.5}\int_0^\infty xe^{-x/6.5}dx\frac{2}{9} + \frac{1}{10}\int_0^\infty xe^{-x/10}dx\frac{1}{9} \\ &= 5\frac{6}{9} + 6.5\frac{2}{9} + 10\frac{1}{9} \approx 5.89 \text{ years.} \end{aligned}$$

Problem 3.7 (4.20 in Stark and Woods)

. We are asked to compute the expected value $E[Y]$ of the received signal. Now

$$\begin{aligned} E[Y] &= E[E[Y|\Theta]] \\ &= \int_{-\infty}^{\infty} E[Y|\theta]f(\theta)d\theta. \end{aligned}$$

Now

$$E[Y|\Theta = \theta] = \theta \quad \text{by inspection of the given conditional Normal density.}$$

Thus

$$E[Y] = \frac{1}{2\pi}\int_0^{2\pi} \theta d\theta = \pi.$$