ECE 4260 Problem Set 2 Solutions

Problem 2.1: (3.5 in Stark and Woods)

(a)

$$f_X(x) = \alpha e^{-\alpha x} u(x)$$

$$Y = e^X \qquad X = \ln(Y) \qquad 1 \leq Y \leq \infty$$

$$f_Y(y) = f_X(\ln(y)) \left| \frac{d \ln y}{dy} \right|$$

$$= \alpha e^{-\alpha(\ln(y))} \frac{1}{y} u(y-1)$$

$$= y^{-1} \alpha e^{-\ln(y^{\alpha})} u(y-1) = y^{-1} \alpha y^{-\alpha} = \alpha y^{-\alpha-1} u(y-1)$$

(b) For this part, only the function g has changed, this time $g(x) = 2x + 3$. We choose to use the indirect method, and find

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P\left[X \leq \frac{y-3}{2}\right] \\ &= F_X\left(\frac{y-3}{2}\right). \end{aligned}$$

Taking the derivative with respect to the free variable y, we get

$$f_Y(y) = \frac{1}{2} f_X\left(\frac{y-3}{2}\right),$$

which, for the exponential distribution with parameter α, becomes

$$\begin{aligned} f_Y(y) &= \frac{1}{2}\alpha e^{-\alpha\frac{y-3}{2}} u(\frac{y-3}{2}) \\ &= \frac{\alpha}{2} e^{-\frac{\alpha}{2}(y-3)} u(y-3). \end{aligned}$$

Problem 2.2 (3.7 in Stark and Woods)

We present two methods to solve this transformation problem:

Method (1): For $y > 0$

$$P[Y \leq y] = P[e^X \leq y] = P[X \leq \ln y] = F_X(\ln y).$$

Hence

$$f_Y(y) = \frac{dF_X(\ln y)}{dy} = \frac{dF_X(\ln y)}{d(lny)}\frac{d(\ln y)}{dy} = \frac{1}{y}f_X(\ln y).$$

For $y \leq 0$

$$P[Y \leq y] = P[e^X \leq y] = P[\phi] = 0.$$

Hence

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\ \sigma y}\exp[-\frac{1}{2}(\frac{\ln y - \mu}{\sigma})^2]u(y).$$

Method (2): A plot of $y = g(x)$ is is $x = \ln y$, for $y > 0$ as given in Fig. 4.

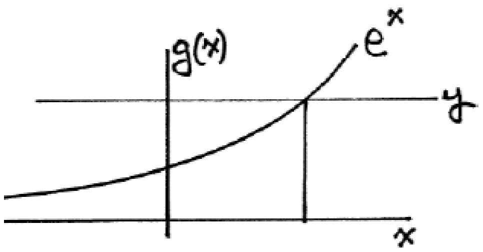

Figure 4:

$$f_Y(y) = \sum_{i=1}^{r}\frac{f_X(x_i)}{|g'(x_i)|} = f_X(\ln y)/e^x|_{x=\ln y}.$$

For $y \leq 0$, there is no real solution to $y - g(x) = 0$; hence for $y \leq 0$, the pdf of Y equals zero there. Combining solutions for both regions of y, we get

$$f_Y(y) = \frac{f_X(\ln y)}{y}u(y).$$

Problem 2.3 (3.10 in Stark and Woods)

Here is the plot of $y = g(x)$ in Fig. 8: $y - g(x) = 0$ is $y = \sqrt{x}$, or $x = y^2$. For $y < 0$, no real solutions exist to $y - g(x) = 0$. Hence

$$f_Y(y) = \frac{f_X(x)}{g'(x)}|_{x=y^2}.$$

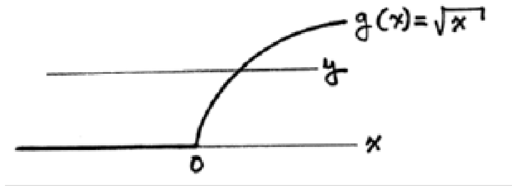

Figure 8:

Now

$$g'(x) = \frac{1}{2}x^{-\frac{1}{2}}|_{x=y^2} = \frac{1}{2y},$$

thus

$$f_Y(y) = \begin{cases} \sqrt{\frac{2}{\pi}} y e^{-\frac{1}{2}y^4}, & y > 0, \\ 0, & y < 0. \end{cases}$$

But at $y = 0$ there is a probability mass, jump in the distribution function, and impulse in the density function. We have

$$\begin{aligned} P[Y = 0] &= P[X \leq 0] \\ &= F_Y(0) = \frac{1}{2}, \end{aligned}$$

thus

$$F_Y(0) = \frac{1}{2} \int_{-\epsilon}^{\epsilon} \delta(y) dy.$$

So, combining results, we have the following answer valid for all y,

$$f_Y(y) = \frac{1}{2}\delta(y) + \sqrt{\frac{2}{\pi}} y e^{-\frac{1}{2}y^4} u(y).$$

Problem 2.4 (3.17 in Stark and Woods)

Since the random variables X and Y are independent, we can find the density of Z via convolution of the two uniform densities f_X and f_Y, thus

$$\begin{aligned} f_Z(z) &= f_X(z) * f_Y(z) \\ &= \int_{-\infty}^{+\infty} f_X(z - x) f_Y(x) dx. \end{aligned}$$

Here, by the problem statement $X : U(-1, +1)$ and hence

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < +1, \\ 0, & \text{else.} \end{cases}$$

Similarly $Y : U(-2, +2)$ and so

$$f_Y(y) = \begin{cases} \frac{1}{4}, & -2 < y < +2, \\ 0, & \text{else.} \end{cases}$$

Computing the convolution graphically, we see that the resulting function f_Z will be constant when the short pulse f_X is completely contained inside the longer pulse f_Y, and that this will occur for $-1 \le z \le +1$, for which the area is easily computed as $\frac{1}{2} \times \frac{1}{4} \times 2 = \frac{1}{4}$. From graphical considerations, we can also easily see that the output function f_Z must be zero when the two pulses do not overlap, and that this will occur for all $|z| > 3$. For $3 \ge |z| > 1$, we then just connect these result together via straight lines, to obtain

$$f_Z(z) = \begin{cases} 0, & z < -3, \\ \frac{1}{8}(z+3), & -3 \le z < -1 \\ \frac{1}{4}, & -1 \le z \le +1, \\ \frac{1}{8}(-z+3), & +1 < z \le +3 \\ 0, & z > +3 \end{cases},$$

which is graphed as Fig. 16.

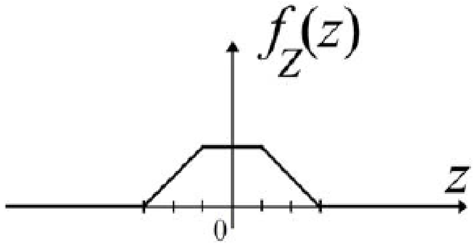

Figure 16:

The problem assignment shifted $f_X(x)$ by 1 to the left and $f_Y(y)$ by two to the right. So the answer is the graph above shifted 1 to the right.

Problem 2.5 (3.22 in Stark and Woods)

If we added two independent random numbers with uniform density over (0.1], then result of the convolution of the two would be a random variable with the pdf shown in fFgure 17. All we need to do to get the desired pdf (Figure 18) is to subtract 1 from the rv derived above,

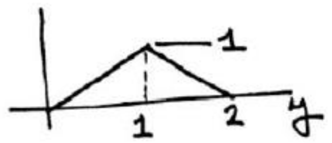

Figure 17:

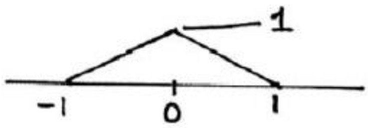

Figure 18: