ECE 4260 Problem Set 12 Solutions

Problem 12.1 (10.7 in Stark and Woods)

We have that $I(T) \triangleq \frac{1}{T}\int_0^T X(t)dt, \qquad T > 0.$

(a)

$$\begin{aligned}
E[I(T)] &= E\left[\frac{1}{T}\int_0^T X(t)dt\right] \\
&= \frac{1}{T}E\left[\int_0^T X(t)dt\right] \\
&= \frac{1}{T}\int_0^T E[X(t)]dt \\
&= \frac{1}{T}\int_0^T \mu_X dt \\
&= \mu_X.
\end{aligned}$$

(b)

$$\begin{aligned}
\sigma^2_{I(T)} &= E[(I(T) - E[I(T)])^2] \\
&= E\left[\left(\frac{1}{T}\int_0^T X(t)dt - \mu_X\right)^2\right] \\
& \quad E\left[\left(\frac{1}{T}\int_0^T (X(t) - \mu_X)\, dt\right)^2\right] \\
&= \frac{1}{T^2}\int_0^T\int_0^T K_{XX}(t-s)dtds \\
&= \frac{1}{T^2}\int_{-T}^T (T - |\tau|)K_{XX}(\tau)d\tau \\
&= \frac{1}{T}\int_{-T}^T \left(\frac{T-|\tau|}{T}\right) K_{XX}(\tau)d\tau,
\end{aligned}$$

where we have made the substitution $\tau \triangleq t - s, \xi \triangleq t + s$, two lines above, and then integrated out in the variable ξ.

Problem 12.2 (10.22 in Stark and Woods)

We have two hypotheses

$$\left.\begin{array}{ll} H_0: & R(t) = W(t) \\ H_1: & R(t) = A + W(t) \end{array}\right\} 0 \le t \le T$$

$$\Lambda = \int_0^T R(t)dt$$

(a) (i) Under hypothesis H_0 :

$$\begin{aligned} E[\Lambda|H_0] &= E\left[\int_0^T R(t)dt|H_0\right] \\ &= E\left[\int_0^T W(t)dt\right] \\ &= E[0] = 0. \end{aligned}$$

(ii) Under hypothesis H_1 :

$$\begin{aligned} E[\Lambda|H_1] &= E\left[\int_0^T R(t)dt|H_1\right] \\ &= E\left[\int_0^T (A + W(t))\, dt\right] \\ & \quad E\left[\int_0^T Adt\right] \\ &= E[AT] = AT. \end{aligned}$$

(b) For the variances, under H_0 :

$$\begin{aligned} \sigma_\Lambda^2 &= E[(\Lambda - \mu_\Lambda)^2|H_0] \\ &= E\left[\left(\int_0^T W(t)dt - \int_0^T \mu_W(t)dt\right)^2 |H_0\right] \\ &= E\left[\left(\int_0^T W(t)dt - 0\right)^2\right] \\ &= E\left[\int_0^T \int_0^T W(t_1)W(t_2)dt_1dt_2\right] \end{aligned}$$

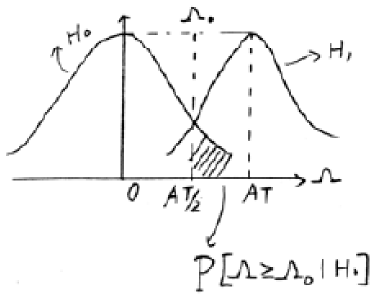
$$
\begin{aligned}
&= \int_0^T \int_0^T \sigma^2 \delta(t_1 - t_2) dt_1 dt_2 \\
&= \sigma^2 T.
\end{aligned}
$$

Under hypothesis H_1, the variance is the same since there is only a shift in the DC value. Thus under each hypothesis $\sigma_\Lambda^2 = \sigma^2 T$.

(c)

$$P[\Lambda \geq \Lambda_0 | H_0] = \frac{1}{\sqrt{2\pi\sigma_\Lambda^2}} \int_{\Lambda_0}^{\infty} e^{-\frac{\alpha^2}{2\sigma_\Lambda^2}} d\alpha,$$

where $\Lambda_0 \triangleq AT/2$. Let $\frac{\alpha}{\sigma_\Lambda} = \eta$, then $d\alpha = \sigma_\Lambda d\eta$. Also $\alpha = \frac{AT}{2}$, which implies $\eta = \frac{AT}{2\sigma_\Lambda}$.



Then

$$
\begin{aligned}
P[\Lambda \geq \Lambda_0 | H_0] &= \frac{\sigma_\Lambda}{\sqrt{2\pi}\sigma_\Lambda} \int_{\frac{AT}{2\sigma_\Lambda}}^{\infty} e^{-\frac{\eta^2}{2}} d\eta \\
&= \frac{1}{2} - \text{erf}\left(\frac{AT}{2\sigma_\Lambda}\right) \\
&= \frac{1}{2} - \text{erf}\left(\frac{AT}{2\sigma\sqrt{T}}\right) \\
&= \frac{1}{2} - \text{erf}\left(\frac{A\sqrt{T}}{2\sigma}\right).
\end{aligned}
$$

Problem 12.3 (10.26 in Stark and Woods)

Define the one-parameter covariance function $K_{XX}(\tau) \triangleq K_{XX}(s+\tau, s) = \sigma^2 \cos\omega_0\tau$, which is seen to be periodic in τ and independent of s. Then since the mean μ_X is constant, we have a WSS periodic random process. For these processes, the Fourier series expansion coefficients are orthogonal, i.e. the Fourier series basis set is also the Karhunen-Loeve basis set. The period of $K_{XX}(\tau)$ is $T \triangleq \frac{2\pi}{\omega_0}$. Hence, any interval of the time axis of width T will do for the expansion.

Problem 12.4 (10.27 in Stark and Woods)

By the equation (10.5-3),

$$\int_{-T/2}^{T/2} K_{XX}(t_1, t_2)\phi_1(t_2)\, dt_2 = \lambda_1 \phi_1(t_1),$$

we have

$$\begin{aligned}
\int_{-T/2}^{T/2} \phi_1(t)\phi_2^*(t)dt &= \frac{1}{\lambda_1}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} K_{XX}(t,t_2)\phi_1(t_2)\phi_2^*(t)dtdt_2 \\
&= \frac{1}{\lambda_1}\int_{-T/2}^{T/2} \phi_1(t_2)\left(\int_{-T/2}^{T/2} K_{XX}^*(t_2,t)\phi_2^*(t)dt\right) dt_2 \\
&= \frac{1}{\lambda_1}\int_{-T/2}^{T/2} \phi_1(t_2)\lambda_2^*\phi_2^*(t_2)dt_2 \\
&= \frac{\lambda_2^*}{\lambda_1}\int_{-T/2}^{T/2} \phi_1(t)\phi_2^*(t)dt \\
&= \frac{\lambda_2}{\lambda_1}\int_{-T/2}^{T/2} \phi_1(t)\phi_2^*(t)dt, \text{ since the } \lambda_i \text{ are real.}
\end{aligned}$$

Because $\lambda_1 \neq \lambda_2$, it must be that the indicated integral is zero.

Problem 12.5 (10.44 in Stark and Woods)

(a) Since the noise process is Gaussian, the K-L expansion ensures independence of the transformed coefficients. The other R_k's are thus independent of R_{k_o}, which is the only one containing the message. Thus

$$P[R_{k_o} \leq r | \{ \text{ all other } R_k' s\}] = P[R_{k_o} \leq r].$$

(b) Since λ_k is the noise mean-square level on basis function (channel) k, we want the smallest λ_k for the signaling channel. So we want $k_o = \infty$. Of course, practical conditions would intercede in reality, forcing a lower finite choice.

Problem 12.6

(a) $\int_0^T \delta(t_1 - t_2)\,\phi_k(t_2)\,dt_2 = \lambda_k \phi_k(t_1)$

Clearly, any orthonormal basis set works. $\lambda_k = 1 \;\forall k$.

(b) $\int_0^T f(t_1) f(t_2) \phi_k(t_2)\,dt_2 = \lambda_k \phi_k(t_1)$

$= f(t_1) \underbrace{\int_0^T f(t_2)\phi_k(t_2)\,dt_2}_{\lambda_k} = \lambda_k \phi_k(t_1)$

$\therefore f(t) = \phi_k(t)$ → only one eigenfunction

Problem 12.7

(a) $E\int_0^T X^2(t)\,dt = E\int_0^T \sum_i X_i \phi_i(t) \sum_j X_j^* \phi_j^*(t)\,dt$

$= E\sum_i |X_i|^2 = \sum_i \lambda_i$

(b) $Y(t) = \int_0^T \phi_1(t)\phi_3^*(u) \sum_i X_i \phi_i(u)\,du$

$+ \int_0^T \phi_2(t)\phi_3^*(u) \sum_i X_i \phi_i(u)\,du$

$= X_3(\phi_1(t) + \phi_2(t))$

Problem 12.8 (10.4 in Stark and Woods)

The random process $X(t)$ is stationary with mean μ_X and covariance function

$$K_{XX}(\tau) = \frac{\sigma_X^2}{1+\alpha^2\tau^2}.$$

(a) We have to show that $R_{XX}(\tau)$ has derivatives up to order two. Because $X(t)$ is stationary, the mean is constant, so that $\mu'_X(t) = 0$, therefore

$$\begin{aligned} \frac{dR_{XX}(\tau)}{d\tau} &= \frac{dK_{XX}(\tau)}{d\tau} \\ &= \frac{-\alpha^2\tau\sigma_X^2}{(1+\alpha^2\tau^2)^2}, \text{ which exists for all finite } \tau. \end{aligned}$$

Next

$$\begin{aligned} \frac{d^2K_{XX}(\tau)}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{-2\alpha^2\tau}{(1+\alpha^2\tau^2)^2}\right)\sigma_X^2 \\ &= \frac{-2\alpha^2(1+\alpha^2\tau^2)^2 + 2\alpha^2\tau\cdot 2\cdot 2\alpha^2\tau(1+\alpha^2\tau^2)}{(1+\alpha^2\tau^2)^4}\sigma_X^2 \\ &= \frac{-2\alpha^2(1+\alpha^2\tau^2)+8\alpha^4\tau^2}{(1+\alpha^2\tau^2)^3}\sigma_X^2 \\ &= \frac{-2\alpha^2(1-3\alpha^2\tau^2)}{(1+\alpha^2\tau^2)^3}\sigma_X^2 \end{aligned}$$

which exists for all τ, and hence, for $\tau = 0$. Therefore the m.s. derivative exists for all finite t.