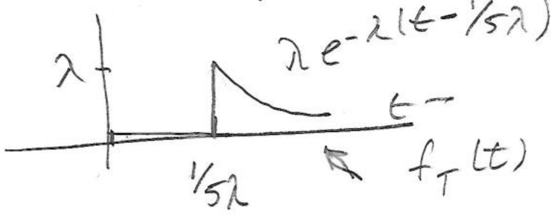
ECE 4260 Problem Set 11 Solutions

Problem 11.1

(a)

→ For every successful capping, there will be on the average $\lambda\left(\frac{1}{5\lambda}\right) = \frac{1}{5}$ bottles destroyed. ∴ ratio of destroyed to not destroyed = 1/5. → Prob(destroyed) = 1/6

→ or: The overall interarrival times between successful cappings follows:



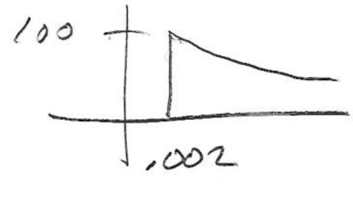
$$E(T) = \frac{1}{\lambda 5} + 1/\lambda = \frac{6}{5\lambda}$$

The rate of successful cappings $= \frac{5\lambda}{6}$. But the underlying rate $= \lambda$ → 1 out of 6 is destroyed

(b)

Here $\lambda = 100$; $\frac{1}{\lambda} = .01$ $\quad .002 = \frac{1}{\lambda 5}$

∴ same construct as above



interfiring time, t

$f_T(t)$

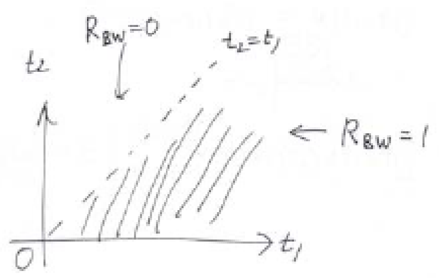
new rate $= \frac{5 \cdot 100}{6} = 83.\bar{3}$

Problem 11.2 (9.50 in Stark and Woods)

(a)

$$\begin{aligned} R_{BW}(t_1,t_2) &= E\left[\int_0^{t_1} W(\tau_1)d\tau_1 W(t_2)\right] \\ &= \int_0^{t_1} R_{WW}(\tau_1,t_2)d\tau_1 \\ &= \int_0^{t_1} \delta(\tau_1 - t_2)d\tau_1 \\ &= u(t_1 - t_2). \end{aligned}$$

Here is a sketch.



(b) Let the two times $t_1, t_2 \geq 0$, then

$$\begin{aligned} R_{BB}(t_1,t_2) &= E\left[\int_0^{t_1}\int_0^{t_2} W(\tau_1)W(\tau_2)d\tau_1 d\tau_2\right] \\ &= \int_0^{t_1}\int_0^{t_2} R_{WW}(\tau_1,\tau_2)d\tau_1 d\tau_2 \\ &= \int_0^{t_1}\int_0^{t_2} \delta(\tau_1-\tau_2)d\tau_1 d\tau_2 \\ &= \int_0^{t_1}\left(\int_0^{t_2} \delta(\tau_1-\tau_2)d\tau_2\right)d\tau_1 \\ &= \int_0^{t_1} u(t_2-\tau_1)d\tau_1 \\ &= \min(t_1,t_2), t_1, t_2 \geq 0. \end{aligned}$$

Problem 11.3 (9.57 in Stark and Woods)

$$\begin{aligned} E[|\widetilde{X}(t)|^2] &= E[|X(t)+U(t)|^2] \\ &= E[|X(t)|^2] + E[|U(t)|^2] \\ &= P + \epsilon \\ &= E[\widetilde{Y}(t)|^2]. \end{aligned}$$

$$\begin{aligned} E[\widetilde{X}(t_1)\widetilde{Y}^*(t_2)] &= E[X(t_1)(Y^*(t_2)+V^*(t_2))] + E[U(t_1)(Y^*(t_2)+V^*(t_2))] \\ &= E[X(t_1)Y^*(t_2)] \\ &= \rho_{XY}(t_1,t_2)P. \end{aligned}$$

So $\rho_{\bar{X}\bar{Y}}(t_1,t_2) = \rho_{XY}(t_1,t_2)\frac{P}{P+\epsilon}$.

Problem 11.4 (9.61 in Stark and Woods)

By equating probability flows, we get the equalities

$$\begin{aligned} \lambda_1 P_1 &= \mu_2 P_2, \\ \lambda_2 P_2 &= \mu_3 P_3, \quad \text{and} \\ \lambda_3 P_3 &= \mu_4 P_4. \end{aligned}$$

From the first equation, $P_2 = \frac{\lambda_1}{\mu_2}P_1$, and then

$$\begin{aligned} P_3 &= \frac{\lambda_2}{\mu_3}P_2 \\ &= \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}P_1, \end{aligned}$$

and

$$\begin{aligned} P_4 &= \frac{\lambda_3}{\mu_4}P_3 \\ &= \frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}P_1. \end{aligned}$$

Using the fact that these four probabilities must also sum to one, i.e. $\sum_i P_i = 1$, we finally get

$$
\begin{aligned}
P_1 &= \frac{1}{1+\frac{\lambda_1}{\mu_2}+\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}+\frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}, \\
P_2 &= \frac{\frac{\lambda_1}{\mu_2}}{1+\frac{\lambda_1}{\mu_2}+\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}+\frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}, \\
P_3 &= \frac{\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}{1+\frac{\lambda_1}{\mu_2}+\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}+\frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}, \text{ and} \\
P_4 &= \frac{\frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}{1+\frac{\lambda_1}{\mu_2}+\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}+\frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}.
\end{aligned}
$$

Problem 11.5 (Problem 9.62 in Stark and Woods)

(a) The probability of leaving state 2 for the first time at time t is zero, since the waiting time is an exponential RV, a continuous RV.

(b)

$$
\begin{aligned}
P_1(t+\delta t) &= (1-\lambda_1\,\delta t)P_1(t)+\mu_2\,\delta t\,P_2(t)+0P_3(t) \\
P_2(t+\delta t) &= +\lambda_1\,\delta t\,P_1(t)-(\lambda_2+\mu_2)\delta t\,P_2(t)+\mu_3\,\delta t\,P_3(t) \\
P_3(t+\delta t) &= 0P_1(t)+\lambda_2\,\delta t\,P_2(t)+-\mu_3\,\delta t\,P_3(t),
\end{aligned}
$$

or

$$
\begin{aligned}
\dot{\mathbf{P}}(t) &= \underbrace{\begin{bmatrix} -\lambda_1 & +\mu_2 & 0 \\ +\lambda_1 & -(\lambda_2+\mu_2) & +\mu_3 \\ 0 & +\lambda_2 & -\mu_3 \end{bmatrix}}_{\triangleq\mathbf{A}}\mathbf{P}(t) \\
&= \mathbf{A}\mathbf{P}(t).
\end{aligned}
$$

(c) We substitute $\exp(\mathbf{A}t)\cdot\mathbf{P}(0)$ into this equation, and then take the term-by-term deriv-

ative of the matrix-exponential series, to obtain

$$
\begin{aligned}
\dot{\mathbf{P}}(t) &= \frac{d}{dt}\left(\sum_{k=0}^{\infty}\frac{1}{k!}(\mathbf{A}t)^k\right)\mathbf{P}(0) \\
&= \left(\sum_{k=1}^{\infty}\frac{1}{k!}k\mathbf{A}^k t^{k-1}\right)\mathbf{P}(0) \\
&= \left(\mathbf{A}\sum_{k'=0}^{\infty}\frac{1}{k'!}\,(\mathbf{A}t)^{k'}\right)\mathbf{P}(0), \quad \text{with} \quad k' \triangleq k-1, \\
&= \mathbf{A}\exp(\mathbf{A}t)\cdot\mathbf{P}(0) \\
&= \mathbf{A}\mathbf{P}(t),
\end{aligned}
$$

as was to be shown.