ECE 4260 Problem Set 10 Solutions

Problem 10.1 (9.5 in Stark and Woods)

By definition of Poisson process with parameter $\lambda(> 0)$, we have

$$\begin{aligned} P_N(n;t) &= P[N(t)=n] \\ &= \frac{\lambda t}{n!}e^{-\lambda t}u[n], \end{aligned}$$

where $u[n]$ is the unit-step function.

(a) Let $t_2 \geq t_1$, then be independent increments property, $N(t_2) - N(t_1)$ and $N(t_1)$ are independent RVs. Also the increment is Poisson distributed with the same parameter λ. Hence

$$\begin{aligned} P_N(n_1,n_2;t_1,t_2) &= P[N(t_1)=n_1, N(t_2)=n_2] \\ &= P[N(t_1)=n_1]P[N(t_2)-N(t_1)=n_2-n_1] \\ &= \frac{\lambda t_1}{n_1!}e^{-\lambda t_1}u[n_1]\frac{\lambda(t_2-t_1)}{(n_2-n_1)!}e^{-\lambda(t_2-t_1)}u[n_2-n_1] \\ &= \frac{\lambda t_1}{n_1!}e^{-\lambda t_1}\frac{\lambda(t_2-t_1)}{(n_2-n_1)!}e^{-\lambda(t_2-t_1)}u[n_1]u[n_2-n_1] \\ &= \frac{\lambda^2 t_1(t_2-t_1)}{n_1!(n_2-n_1)!}e^{-\lambda t_2}u[n_1]u[n_2-n_1]. \end{aligned}$$

(b) Since the t_i's are increasing, $N(t)$ has independent increments, which can be recursively applied to conclude

$$\begin{aligned} P_N(n_1,n_2,...,n_K;t_1,t_2,...,t_K) &= P[N(t_1)=n_1, N(t_2)=n_2,...,N(t_K)=n_K] \\ &= P[N(t_1)=n_1]P[N(t_2)-N(t_1)=n_2-n_1]\cdots \\ &\qquad \cdots P[N(t_K)-N(t_{K-1})=n_K-n_{K-1}] \\ &= \frac{\lambda^K t_1(t_2-t_1)\cdots(t_K-t_{K-1})}{n_1!(n_2-n_1)!\cdots(n_K-n_{K-1})!}e^{-\lambda t_K}u[n_1]u[n_2-n_1]\cdots u[n_K-n_{K-1}]. \end{aligned}$$

Problem 10.2 (9.6 in Stark and Woods)

(a) Use property (3) for $t_1 = 0$ and $t_2 = t$. Then by the property (1), $N(0) = 0$. So, (3) becomes:

$$P[N(t)=n] = \frac{\left(\int_0^t \lambda(s)ds\right)^n}{n!}e^{-\left(\int_0^t \lambda(s)ds\right)^n} \quad \text{for } n \geq 0.$$

...

Then since $N(t)$ is Poisson distributed, we recognize the mean as

$$\mu_N(t) = \int_0^t \lambda(s)ds, \quad t \geq 0.$$

(b) Take $t_2 \geq t_1 \geq 0$, and write $E[N(t_1)N(t_2)] = E[N(t_1)[N(t_1) + (N(t_2) - N(t_1))]]$. Then using the linearity of the expectation operator E and the independent increments property (2), we get

$$\begin{aligned} R_N(t_1, t_2) &\triangleq E[N(t_1)N(t_2)] \\ &= E[N^2(t_1)] + E[N(t_1)]E[N(t_2) - N(t_1)]. \end{aligned}$$

We then recognize the first term on the rhs as the second moment of the Poisson, therefore

$$E[N^2(t_1)] = \int_0^t \lambda(s)ds + \left(\int_0^t \lambda(s)ds\right)^2.$$

Now $E[N(t_2) - N(t_1)] = \int_{t_1}^{t_2} \lambda(s)ds$, and from part (a) $E[N(t_1)] = \int_0^t \lambda(s)ds$, so, putting these together, we get

$$\begin{aligned} R_N(t_1, t_2) &= E[N^2(t_1)] + E[N(t_1)]E[N(t_2) - N(t_1)], \quad t_2 \geq t_1 \geq 0, \\ &= \int_0^{t_1} \lambda(s)ds + \left(\int_0^{t_1} \lambda(s)ds\right)^2 + \left(\int_0^{t_1} \lambda(s)ds\right)\left(\int_{t_1}^{t_2} \lambda(s)ds\right) \\ &= \left(\int_0^{t_1} \lambda(s)ds\right)\left(1 + \int_0^{t_2} \lambda(s)ds\right), \quad t_2 \geq t_1 \geq 0. \end{aligned}$$

For the general case, from the symmetry of the correlation function $R_N(t_1, t_2) \triangleq E[N(t_1)N(t_2)]$, we can write

$$R_N(t_1, t_2) = \left(\int_0^{\min(t_1,t_2)} \lambda(s)ds\right)\left(1 + \int_0^{\max(t_1,t_2)} \lambda(s)ds\right), \quad t_1, t_2 \geq 0.$$

(c) We have to show properties (1), (2), and (3):

(1) $N_u(0) \triangleq N(t(0)) = N(0) = 0$. ✓

(2) Let $\tau_1 \leq \tau_2 \leq \tau_3 \leq \cdots \leq \tau_k$. Then since $t(\tau)$ is monotone increasing (since it is the integral of a positive $\lambda(s)$), we have $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_k$ where $t_i \triangleq t(\tau_i)$. Thus $N_u(\tau_i) \triangleq N(t(\tau_i)) = N(t_i)$. So, by definition, $N(t_1), N(t_2) - N(t_1), ...,$ $N(t_k) - N(t_{k-1})$ are jointly independent. But $N_u(\tau_i) - N_u(\tau_{i-1}) = N(t_i) - N(t_{i-1})$, so the $N_u(\tau)$ process also has independent increments.

(3) Since $N_u(\tau_2) - N_u(\tau_1) = N(t_2) - N(t_1)$ with mean value

$$\begin{aligned} \int_{t_1}^{t_2} \lambda(s)ds &= \int_0^{t_2} \lambda(s)ds - \int_0^{t_1} \lambda(s)ds \\ &= \tau_2 - \tau_1 \end{aligned}$$

which means that $N(\tau)$ has λ parameter equal to 1.

Problem 10.3 (9.7 in Stark and Woods)

The Poisson PMF is given as

$$P_N(n;t) = \frac{\left(\int_0^t \lambda(\nu)d\nu\right)^n}{n!} e^{-\left(\int_0^t \lambda(\nu)d\nu\right)^n} u[n].$$

(a)

$$\begin{aligned} \mu_N(t) &= E[N(t)] \\ &= \int_0^t \lambda(\nu)d\nu, \qquad t \geq 0, \\ &= \int_0^t (1+2\nu)d\nu \\ &= \nu + \nu^2\big|_0^t \\ &= t + t^2, \qquad t \geq 0. \end{aligned}$$

(b)Let $t_2 \geq t_1 \geq 0$, then

$$N(t_1)N(t_2) = N(t_1)[N(t_1) + (N(t_2) - N(t_1))],$$

so

$$R_N(t_1, t_2) = E[N^2(t_1)] + \mu_N(t_1)\left(\mu_N(t_2) - \mu_N(t_1)\right).$$

Now

$$\begin{aligned} E[N^2(t)] &= \int_0^t \lambda(\nu)d\nu + \left(\int_0^t \lambda(\nu)d\nu\right)^2 \\ &= \left(t+t^2\right) + \left(t+t^2\right)^2. \end{aligned}$$

So for $t_2 \geq t_1$,

$$\begin{aligned} R_N(t_1,t_2) &= \left(t_1+t_1^2\right) + \left(t_1+t_1^2\right)^2 + \left(t_1+t_1^2\right)\left(\left(t_2+t_2^2\right) - \left(t_1+t_1^2\right)\right) \\ &= \left(t_1+t_1^2\right) + \left(t_1+t_1^2\right)\left(t_2+t_2^2\right). \end{aligned}$$

In general, we have

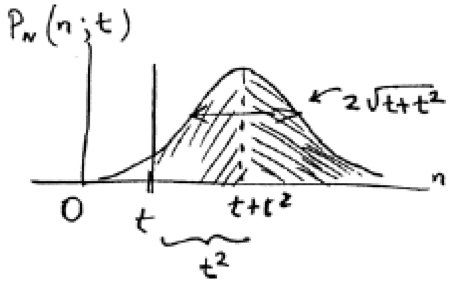
$$R_N(t_1, t_2) = \left[\min(t_1, t_2) + (\min(t_1, t_2))^2\right]\left[1 + \max(t_1, t_2) + (\max(t_1, t_2))^2\right].$$

(c)

$$\begin{aligned} P[N(t) \geq t] &= \sum_{n \geq \lceil t \rceil} P_N(n;t) \\ &= \sum_{n \geq \lceil t \rceil} \frac{(t+t^2)^n}{n!} e^{-(t+t^2)}, \quad t \geq 0. \end{aligned}$$

(d) Use the CLT with $\mu_N(t) = t + t^2$ and $\sigma_N^2 = t + t^2$ to yield

$$\begin{aligned} P[N(t) &\geq t] \approx \frac{1}{2} + \text{erf}\left(\frac{t^2}{\sqrt{t+t^2}}\right) \\ &\approx \frac{1}{2} + \text{erf}(t). \end{aligned}$$



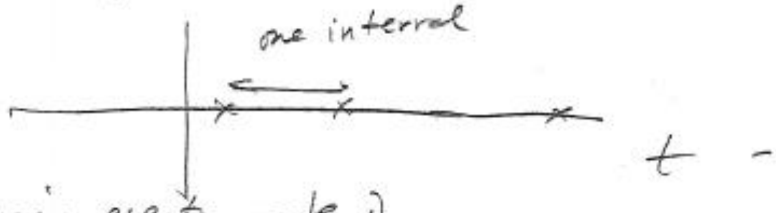
We remember

$$\begin{aligned} \text{erf}(x) &= \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}v^2} dv \\ &= P[X_{SN} \leq x] \quad \text{for} \quad X_{SN} : N(0,1). \end{aligned}$$

Problem 10.4

(a) Yes. No particular place is favored in time. We will see this in part (b) also

(b) $R_{XX}(t_1, t_2) = ?$



"x" = Poisson events, rate λ.

$R_{XX}(t_1, t_2) = E(X(t_1)\, X(t_2)) =$

$1 \cdot \text{Prob}(t_1 \,\&\, t_2$ in same interval$)$

$+\, 0 \cdot \text{Prob}(t_1 \,\&\, t_2$ in different intervals$)$

$\text{Prob}(t_1 \,\&\, t_2$ are in same interval$) =$

prob$(0$ events over $(t_1, t_2)) = e^{-\lambda(t_2 - t_1)}$ if $t_2 > t_1$

$\therefore\; R_{XX}(t_1, t_2) = e^{-\lambda|t_2 - t_1|} = e^{-\lambda|\tau|}$

(c) $R_{YY}(\tau) = R_{XX}(\tau) \cdot a^2 = a^2 e^{-\lambda|\tau|}$

(d) $R_{VV}(\tau) = \frac{1}{4}\left(R_{XX}(\tau) + 1\right) = \frac{1}{4} e^{-\lambda|\tau|} + \frac{1}{4}$

(e) this construct, in contrast to the construct in the book, allows the value of $X(t)$ to stay the same or change at new intervals. $\therefore$ The rate of changes is $\frac{1}{2}$ of that when values must change. → equivalent to a new rate $= 2\lambda$ for the book →

$R(\tau) = e^{-2\lambda|\tau|}$

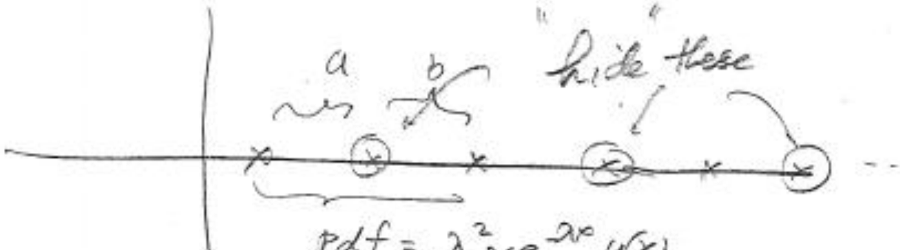
Problem 10.5

Consider a Poisson process, rate λ.

"x" events.

PDF for interarrival times $= \lambda e^{-\lambda x} u(x)$

If we "hide" every other event, we would get a renewal process with interarrival time pdf $= \lambda^2 x e^{-\lambda x} u(x)$



pdf $= \lambda^2 x e^{-\lambda x} u(x)$

Going back to the previous problem logic.

$R_{XX}(t_1, t_2) = $ Prob(t_1 & t_2 are in the same interval)

Given that the interarrival times are <u>not</u> exponential, the system is no longer memoryless. But there are only 2 states. Referring to the figure above, if t_1 is in an interval, type "a", where it is after an "observed" event, but before a "hidden" event, prob(t_1 & t_2 same interval) = prob(0 or 1 poisson events occur before t_2) $= e^{-\lambda\tau} + \lambda\tau e^{-\lambda\tau}$. If t_1 is in an interval type "b", where it is after a "hidden" event, prob(t_1 & t_2 are same interval) = Prob(0 poisson events occur before t_2) $= e^{-\lambda\tau}$. Types "a" & "b" are equally likely,

$$R_{XX}(\tau) = \frac{1}{2} e^{-\lambda|\tau|} + \frac{1}{2}\left(e^{-\lambda|\tau|} + \lambda|\tau| e^{-\lambda|\tau|}\right)$$

$$= e^{-\lambda|\tau|}\left(1 + \frac{\lambda|\tau|}{2}\right)$$