ECE 4260 Problem Set 1 Solutions

Problem 1.1: (2.21 in Stark and Woods)

The random variables X and Y have joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{4}x^2(1-y), & 0 \leq x \leq 2, 0 \leq y \leq 1, \\ 0, & \text{else.} \end{cases}$$

(a) To find $P[X \leq 0.5]$, we start with

$$\begin{aligned}
P[X \leq 0.5] &= \int_{-\infty}^{0.5} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy \\
&= \int_{0}^{0.5} \int_{0}^{1} \frac{3}{4}x^2(1-y)dxdy \\
&= \frac{3}{4}\left(\int_{0}^{0.5} x^2 dx\right)\left(\int_{0}^{1}(1-y)dy\right) \\
&= \frac{3}{4}\left(\frac{x^3}{3}|_0^{0.5}\right)\left((y-\frac{y^2}{2})|_0^1\right) \\
&= \frac{3}{4}\frac{1}{24}(1-\frac{1}{2}) = \frac{1}{64}.
\end{aligned}$$

(b) By definition

$$\begin{aligned}
F_Y(0.5) &= P[Y \leq 0.5] \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{0.5} f_{X,Y}(x,y)dxdy \\
&= \int_{0}^{2} \int_{0}^{0.5} \frac{3}{4}x^2(1-y)dxdy \\
&= \frac{3}{4}\left(\frac{x^3}{3}|_0^2\right)\left((y-\frac{y^2}{2})|_0^{0.5}\right) \\
&= \frac{3}{4}\frac{8}{3}\left(\frac{1}{2}-\frac{1}{8}\right) = \frac{3}{4}.
\end{aligned}$$

(c) To find $P[X \le 0.5|Y \le 0.5]$, we note that X and Y are independent random variables, so the answer is the same as in part a), namely $P[X \le 0.5|Y \le 0.5] = P[X \le 0.5] = \frac{1}{64}$. However, we can also calculate directly,

$$\begin{aligned} P[X &\le 0.5|Y \le 0.5] = \frac{P[X \le 0.5, Y \le 0.5]}{P[Y \le 0.5]} \\ &= \int_0^{0.5} \int_0^{0.5} \frac{3}{4} x^2 (1-y) dx dy / \left(\frac{3}{4}\right) \\ &= \frac{3}{4} \frac{1}{24} \left(\frac{1}{2} - \frac{1}{8}\right) / \left(\frac{3}{4}\right) = \frac{1}{64}. \end{aligned}$$

(d) Here, we can note again that X and Y are independent random variables for the given joint pdf, and thus

$$\begin{aligned} P[Y &\le 0.5|X \le 0.5] = P[Y \le 0.5] \\ &= \frac{3}{4} \quad \text{from part b).} \end{aligned}$$

Problem 1.2 (2.22 in Stark and Woods)

To check for independence, we need to look at the marginal pdfs of X and Y. How do we find the pdf's? We can use the property that the pdf must integrate to 1. Say $f_X(x) = Ae^{-\frac{1}{2}(\frac{x}{3})^2}u(x)$, and $\int_0^\infty f_X(x)dx = 1$, we find $A = \frac{2}{3\sqrt{2\pi}}$. Similarly, $f_Y(y) = Be^{-\frac{1}{2}(\frac{x}{2})^2}u(y)$,

and $\int_0^\infty f_Y(y)dy = 1$, so $B = \frac{2}{2\sqrt{2\pi}}$. Multiplying the two marginal pdfs, we see that the product is indeed equal to joint pdf; i.e., $f_X(x)f_Y(y) = f_{X,Y}(x,y)$. Therefore, X and Y are independent random variables; their joint probability factors and hence $P[0 < X \le 3, 0 < Y \le 2] = P[0 < X \le 3]P[0 < Y \le 2]$. Thus

$$\begin{aligned} P[0 < X \le 3] &= \int_{-3}^{3} \frac{2}{3\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x}{3})^2} dx \\ &= 2 \times \frac{2}{3\sqrt{2\pi}} \int_0^3 e^{-\frac{1}{2}(\frac{x}{3})^2} dx = 2erf(1), \end{aligned}$$

$$\begin{aligned} P[0 < Y \le 2] &= \int_{-2}^{2} \frac{2}{2\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y}{2})^2} dy \\ &= 2 \times \frac{2}{2\sqrt{2\pi}} \int_0^2 e^{-\frac{1}{2}(\frac{y}{2})^2} dy = 2erf(1). \end{aligned}$$

So

$$\begin{aligned} &P[0 < X \le 3, 0 < Y \le 2] = P[0 < X \le 3]P[0 < Y \le 2] \\ &= 2\,\mathrm{erf}(1) \times 2\,\mathrm{erf}(1) = 4\,\mathrm{erf}(1)^2 \doteq 0.466. \end{aligned}$$

Problem 1.3 (2.25 in Stark and Woods)

We use Bayes' formula for pdf's:

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}.$$

We have

$$f_X(x) = \frac{1}{2}\text{rect}(\frac{x}{2}).$$

Then

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx$$
$$= \int_{-1}^{1} \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y-x)^2}{2\sigma^2}\right] dx.$$

Let $\xi = \frac{x-y}{\sigma}$, then $d\xi = \frac{dx}{\sigma}$ and we obtain

$$f_Y(y) = \frac{1}{2}\int_{\frac{-1-y}{\sigma}}^{\frac{1-y}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2} d\xi = \frac{1}{2}\left[\text{erf}(\frac{1-y}{\sigma}) - \text{erf}(\frac{-1-y}{\sigma})\right].$$

But $erf(x) = -erf(-x)$, hence

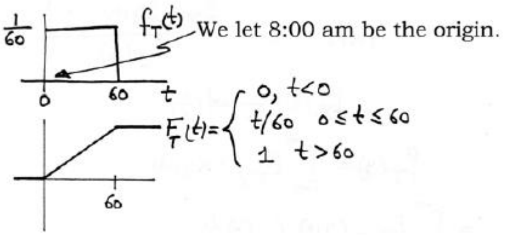
$$f_Y(y) = \frac{1}{2}\left[\text{erf}(\frac{1+y}{\sigma}) - \text{erf}(\frac{y-1}{\sigma})\right].$$

Then finally

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} = \frac{\frac{1}{\sqrt{2\pi}\ \sigma}\exp[-\frac{(y-x)^2}{2\sigma^2}]\text{rect}(\frac{x}{2})}{\text{erf}(\frac{1+y}{\sigma}) - \text{erf}(\frac{y-1}{\sigma})}.$$

Problem 1.4 (2.27 in Stark and Woods)

Let T be the prof's arrival time.



$$
\begin{aligned}
P[A] &= P[T > 30] = 1 - F_T(30) \\
P[B] &= P[T \le 31] = F_T(31) \\
P[AB] &= P[30 < T \le 31] = F_T(31) - F_T \\
P[B|A] = \frac{P[AB]}{P[A]} &= \frac{F_T(31) - F_T(30)}{1 - F_T(30)} \\
&= \frac{\frac{31-30}{60}}{\frac{60-30}{60}} = \frac{1}{30}.
\end{aligned}
$$

$$
\begin{aligned}
P[A|B] &= \frac{P[AB]}{P[B]} = \frac{F_T(31) - F_T(30)}{F_T(31)} \\
&= \frac{\frac{31-30}{60}}{\frac{31}{60}} = \frac{1}{31}.
\end{aligned}
$$

Problem 1.5 (2.36 in Stark and Woods).

The problem was modified to change 0.95 to 0.6825. Substitute 0.6826 for 0.95 in the last lines, giving: –ln(1-0.6826) = 1.148.; c^2 = 2.296; c=1.515.

$$
\begin{aligned}
P[X^2 + Y^2 \le c^2] &= \iint_{(x,y)\,:x^2+y^2 \le c^2} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dxdy, \quad \text{transform to polar coordinates} \\
&= \frac{1}{2\pi} \int_0^c \int_0^{2\pi} e^{-\frac{1}{2}r^2} r dr d\theta, \quad \text{with } r = \sqrt{x^2+y^2} \text{ and } dxdy = rdrd\theta, \\
&= \int_0^c e^{-\frac{1}{2}r^2} r dr, \quad \text{let } u \triangleq \frac{1}{2}r^2, \quad \text{then } du = rdr, \\
&= \int_0^{c^2/2} e^{-u} du \\
&= 1 - e^{-c^2/2} = 0.95.
\end{aligned}
$$

Thus we need

$$
\begin{aligned}
\frac{c^2}{2} &= \ln \frac{1}{1 - 0.95} = \ln 20 \simeq 3, \\
c &\simeq \sqrt{6} = 2.45.
\end{aligned}
$$

Problem 1.6 (2.37 in Stark and Woods)

Note: The problem was modified in parts (b) and (d) to exchange X with Y. By symmetry, simply exchange the variables in the answer

(a) Since the area of this square with side $\sqrt{2}$ is 2, constant joint density $f_{X,Y}$ must take on value $\frac{1}{2}$ to be properly normalized, thus $A = \frac{1}{2}$.

(b) We can see four regions for the y values in evaluating

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y)dy.$$

These regions are $x \leq -1, -1 < x < 0, 0 \leq x < 1$,and $x \geq 1$. Now, the first and last of these regions gives the trivial result $f_X(x) = 0$. For $0 \leq x < 1$, we get

$$f_X(x) = \int_{x-1}^{1-x} \frac{1}{2}dy = \frac{1}{2}(1 - x - x + 1) = 1 - x.$$

Similarly for $-1 < x < 0$, we get

$$f_X(x) = \int_{-x-1}^{1+x} \frac{1}{2} dy = \frac{1}{2}(1 + x + x + 1) = 1 + x.$$

Combining these regions we finally get

$$f_X(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & \text{else}. \end{cases}$$

(c) If X is close to 1, then we see that Y must be close to 0. This suggests dependen between X and Y. To be sure we can use the result of part b together with the symmet of the joint density to check whether $f_{X,Y} = f_X f_Y$ or not. By symmetry of $f_{X,Y}$ must also be that

$$f_Y(y) = \begin{cases} 1 - |y|, & |y| < 1, \\ 0, & \text{else}. \end{cases}$$

Now the product of these two triangles $(1 - |x|)(1 - |y|) \neq \frac{1}{2}$ on $\text{supp}(f_{X,Y})$, so t random variables are definitely dependent. (The *support* of a function $f(x)$ is the set domain values $\{x | f(x) \neq 0\}$ and is written as $\text{supp}\{f\}$.)

(d) We start with the definition and then plug in our result from part b:

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \begin{cases} \frac{0.5}{1-|x|}, & 0 \leq |x| + |y| < 1, \\ 0, & \text{otherwise in } \{|x| < 1\}, \\ \times, & |x| \geq 1. \end{cases} \end{aligned}$$

Note that the conditional density is not defined for $\{|x| \geq 1\}$.

For parts (b) and (d), please note that by symmetry, the variables X and Y can be exchanged without change to the answer.