GEORGIA INSTITUTE OF TECHNOLOGY

School of Electrical and Computer Engineering

Quiz # 2

Date: October 30, 2013 **Course:** ECE 4260A

Name: SOLUTION

Last, First

- Closed book, closed notes, two handwritten sheets allowed.
- None of the problems require involved calculations. Reconsider your approach before doing something tedious.

mean: 73.1

median: 70.5

$\hat{\sigma}$: 18.7

$\sqrt{\hat{\rho}_n}$: 0.863

Problem	*Score*
1	40
2	60
Total	100

Problem 1:

Let $X(t)$ be a continuous-time Gaussian random process. The process is sampled at $t = 0$, $t = 2$ and $t = 4$ yielding three random variables which we arrange in a random vector $\underline{X}$. The mean vector and covariance matrix for these three RVs are listed below:

$$\underline{\mu}_X = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$K_X = \begin{pmatrix} 10 & 7 & 1 \\ 7 & 11 & 2 \\ 1 & 2 & 12 \end{pmatrix}$$

Also, the eigenvectors and eigenvalues for K_X are:

$\underline{e}_1 = [0.721, 0.2775, 0.6341]^T$; $\underline{e}_2 = [-0.6879, 0.1866, 0.7014]^T$; $\underline{e}_3 = [0.0764, -0.9424, 0.3256]^T$
$\lambda_1 = 3.4338$; $\lambda_2 = 11.3096$; $\lambda_3 = 18.2565$

10 (a) Is it possible for this random process to be wide-sense stationary? Justify your answer.

No! Clearly the mean of $X(0)$ is not the mean of $X(2)$, etc. Many other reasons also.

To possibly be WSS, $\mu = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$ $K_X = \begin{pmatrix} b & c & d \\ c & b & c \\ d & c & b \end{pmatrix}$

where a, b, c, d are constants

10 (b) Suppose $Y = X(0) + X(2) + X(4)$. Find the mean and variance of Y.

$\bar{Y} = 3 + 1 + 2 = 6$

$\sigma_Y^2 = \underbrace{(1\ 1\ 1)\, K_X \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{sum of terms in } K_X} = 53$

10 (c) Find $E[(\underline{X} - \underline{\mu}_X)^T(\underline{X} - \underline{\mu}_X)]$ =

$$\text{Trace}\{K_{\underline{X}}\} = 33$$

10 (d) Let $\underline{W} = [X(0), X(4)]^T$. Find the PDF for $\underline{W}$.

X(t) was described as Gaussian

$$\therefore \begin{pmatrix} X[0] \\ X[4] \end{pmatrix} \sim N\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 10 & 1 \\ 1 & 12 \end{pmatrix} \right)$$

5 (e) (Bonus) Let $V = \underline{a}^T\underline{X}$. Also constrain $\underline{a}$ to have length 1. Find $\underline{a}$ that maximizes the variance of V. Also find this maximum variance.

maximum eigenvalue & its eigenvector

$$\underline{a}^T = \underline{e}_3^T = [0.076, -0.9424, 0.3256]$$

$$\sigma_V^2 = \lambda_3 = 18.2565$$

Problem 2

(a) A discrete-time random process has sample functions of the form: $X[n] = A$ where A is a Gaussian random variable of mean 2 and variance 1.

4 (i) Find the mean of $X[n]$.

$$E\left[X[n]\right] = E(A) = 2$$

4 (ii) Find the power in $X[n]$.

$$\overline{X^2[n]} = \text{power} = 2^2 + 1 = 5$$

4 (iii) Find $R_X[m_1, m_2]$, the autocorrelation function of $X[n]$.

$$R_{XX}[m_1, m_2] = A^2 = 5 \quad \forall \; m_1, m_2$$

4 (iv) Is $X[n]$ deterministic or not? Justify your answer.

Yes. Make one observation of one point of a sample function. All other values of $x(n)$ a determined.

(b) $W[n]$ is zero mean discrete-time WSS white noise with spectral height of 1.

4 (i) What is its power?

$$\text{power} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\, d\omega = 1$$

$W[n]$ is put through an ideal lowpass filter with gain 1 and cutoff $\pi/3$ radians. The filter's corresponding impulse response is $\frac{\sin\frac{\pi}{3}n}{\pi n}$. The output is $Y[n]$.

4 (ii) Find $R_{YY}[m]$. $\Rightarrow$

$$\underbrace{R_{WW}[m]}_{\delta[m]} * \underbrace{h[m] * h[-m]}_{h[m]} = h[m] = \frac{\sin\frac{\pi}{3}m}{\pi m}$$

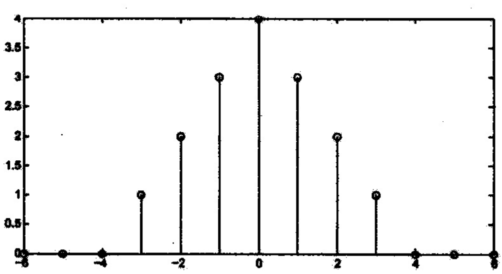
(iii) Find the variance of $Y[n]$.

$$\sigma_Y^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{YY}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} d\omega = 1/3$$

(iv) Find $R_{YW}[m]$

$$R_{YW}[m] = R_{WW}[m] * h[m] = \frac{\sin\frac{\pi}{3}m}{\pi m}$$

(c) $G[n]$ is stationary, zero mean, and has autocorrelation function as sketched below:



4 (i) If $G[n]$ is Gaussian, find the joint PDF for $[G[1], G[2]]$

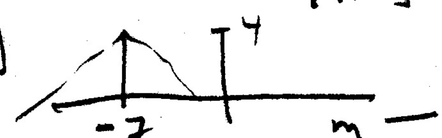
$$\begin{pmatrix} G(1) \\ G(2) \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\right]$$

4 (ii) Find $\frac{1}{2\pi}\int_{-\pi}^{\pi} S_G(\omega)d\omega$ where $S_G(\omega)$ is the power spectral density of $G[n]$.

$$= R_{GG}[0] = 4$$

4 (iii) $G[n]$ is input to a system with impulse response equal to $\delta[n-7]$ (i.e., a delay by 7). The output is $F[n]$. Find $R_{GF}[m]$. $= R_{GG}[m] * h[-m] = R_{GG}[m] * \delta[m+7]$

$$= R_{GG}[m+7]$$



(d) $J[n]$ and $K[n]$ are independent, zero mean stationary random processes. $R_{JJ}[m] = 2e^{-|m|}$; $R_{KK}[m] = 3e^{-(m^2)}$.

4 (i) Find the power in $3J[n] - 2K[m]$.

$= 9 \cdot$ power in J + $4 \cdot$ power in K

$18 + 12 = 30$

4 (ii) Let $L[m] = J[m] + K[m]$. Find $R_{LJ}[m]$. $= E(L[n]J[n+m])$

$$= E[(J[n] + K[n])(J[n+m])] = R_{JJ}[m] = 2e^{-|m|}$$

4 (iii) $J[n]$ was obtained by passing unit spectral height white noise through a filter. Find a possible impulse response for that filter.

many ways to do this. $\Phi_{JJ}(z)$ has poles at e and e^{-1}

we only need one pole. we also need a scale.

In general, $a^n u(n) * a^{-n} u(-n) = \text{scale} \cdot a^{|n|}$

$\text{scale} = \sum_{n=0}^{\infty} a^{2n} = \frac{1}{1-a^2} \rightarrow h(n) = \sqrt{2}\sqrt{1-a^2}\, a^n u(n)$

$a = e^{-1}$ $= \sqrt{2}\sqrt{1-e^{-2}}\, e^{-n} u(n)$

is one choice