GEORGIA INSTITUTE OF TECHNOLOGY

School of Electrical and Computer Engineering

Quiz # 1

Date: September 23, 2013

Course: ECE 4260A

Name: SOLUTION

Last, First

- Closed book, closed notes, one handwritten sheet allowed.
- None of the problems require involved calculations. Reconsider your approach before doing something tedious.

Problem	*Score*
1	50
2	50
Total	100

mean: 70.1
median: 71
st.dev: 17.0

Problem 1

Consider a pair of jointly Gaussian random variables X & Y. Below are some key statistics:

$$
\begin{aligned}
E(X) &= 0\\
E(Y) &= 2\\
E(X^2) &= 9\\
E(Y^2) &= 8\\
E(XY) &= -1
\end{aligned}
$$

Note the probability density function formula below:

$$\text{If } X \sim N(a, \sigma^2) \text{ then } f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}(x-a)^2/\sigma^2\right]$$

(a) Find the marginal densities of X and Y.

$X \sim N(0,9)$ $\qquad$ $Y \sim N(2,4)$

$\sigma_X^2 = \overline{X^2} - \overline{X}^2$ $\qquad$ $\sigma_y^2 = \overline{Y^2} - \overline{Y}^2 = 8 - 4 = 4$

(b) Find the covariance matrix for the vector $[X, Y]^T$.

$$C = \begin{bmatrix} \sigma_X^2 & E[(X-\bar{X})(Y-\bar{Y})] \\ (\,) & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}$$

(c) Find the probability that $-1 \leq X \leq 1$. Express this in terms of $\Phi(\alpha)$, the cumulative distribution function of a zero mean, unit variance Gaussian random variable.

$$\text{Prob}\left[-1 \leq X \leq 1\right] = \Phi\left(\tfrac{1}{3}\right) - \Phi\left(-\tfrac{1}{3}\right) = 2\Phi\left(\tfrac{1}{3}\right) - 1$$

(d) An average is taken of ten, independent experimental values of X. Find the PDF of this average.

$$M = \frac{1}{10}\sum_{i=1}^{10} X_i \; ; \quad \overline{M} = \overline{X} = 0$$

$$\sigma_M^2 = \frac{10\,\sigma_X^2}{(10)^2} = \frac{\sigma_X^2}{10}$$

$$\boxed{M \sim N\left(0, \frac{9}{10}\right)}$$

(e) Let $W = 3X - 5Y$; $Z = 3Y$. Find the joint PDF of W and Z.

$\begin{pmatrix} W \\ Z \end{pmatrix}$ are also JGRV's

$$\begin{pmatrix} W \\ Z \end{pmatrix} \sim N\left[\begin{pmatrix} 3, & -5 \\ 0, & 3 \end{pmatrix}\begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix}; \begin{pmatrix} 3 & -5 \\ 0 & 3 \end{pmatrix} C' \begin{pmatrix} 3 & 0 \\ -5 & 3 \end{pmatrix}\right]$$

$$= N\left(\begin{pmatrix} -10 \\ 6 \end{pmatrix}; \begin{pmatrix} 211 & -69 \\ -69 & 36 \end{pmatrix}\right)$$

Problem 2

For this problem, you may find the following formula useful:

$$\int_0^\infty x^n \exp[-ax]dx = \frac{n!}{a^{n+1}}$$

Al and Bo are engaged in a timed tricycle race.

For Al, the probability density function for the time (in minutes) required to finish is:

$f_X(x) = \lambda^2 x e^{-\lambda x} u(x)$.

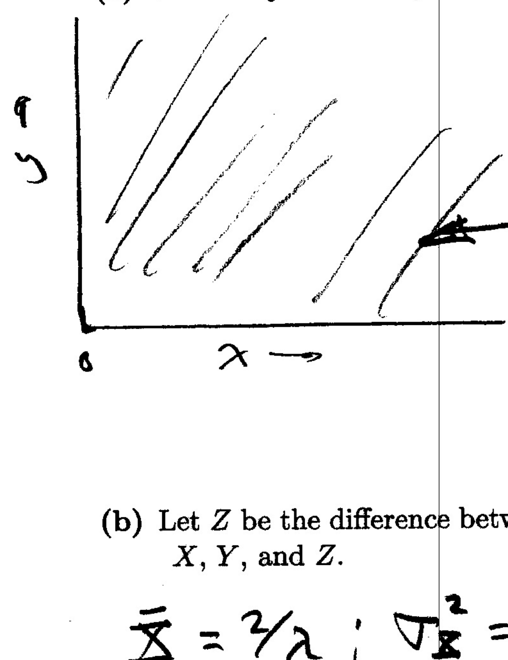
For Bo, the probability density function for the time required to finish is:

$f_Y(y) = \lambda e^{-\lambda y} u(y)$.

The notation $u(x)$ signifies the unit step function.

The times are statistically independent of each other.

(a) Find the joint density function for X and Y. A drawing will be helpful.



Since X & Y are statistically independent,

$f_{XY}(x,y) = f_X(x)\, f_Y(y) =$

$\lambda^3 x e^{-\lambda(x+y)}$

$0 \le y \le \infty$

$0 \le x \le \infty$

(b) Let Z be the difference between the two times: $Z = X - Y$. Find the means and variances for X, Y, and Z.

$\bar{X} = 2/\lambda \,;\; \sigma_X^2 = 2/\lambda^2$

$\bar{Y} = 1/\lambda \,;\; \sigma_Y^2 = 1/\lambda^2$

$\bar{Z} = 1/\lambda \,;\; \sigma_Z^2 = 3/\lambda^2$

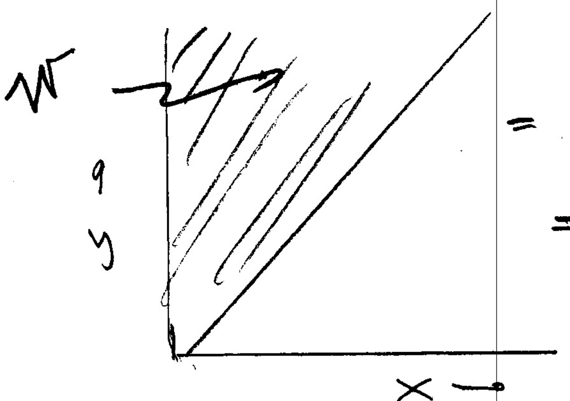
$X \sim$ 2nd order Erlang

$Y \sim$ 1st order Erlang = Exponential

$\bar{Z} = \bar{X} - \bar{Y}$

$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$

(c) Let W represent the event {Al beats Bo} (i.e., $X < Y$). Find Prob(W).



$$Pr(W) = \int_0^\infty \int_x^\infty f_{XY}(x,y)\, dy\, dx$$
$$= \int_0^\infty \lambda^2 x e^{-\lambda x} \int_x^\infty \lambda e^{-\lambda y}\, dy \quad dx$$
$$= \int_0^\infty \lambda^2 x e^{-2\lambda x}\, dx = \frac{1}{4}\int_0^\infty (2\lambda)^2 x e^{-(2\lambda)x}\, dx$$
$$= \boxed{1/4}$$

X represents the time until the 2nd arrival in a Poisson process of rate λ. Y represents the time until the 1st arrival in an identical, independent Poisson process. $X < Y = W$ is the event where the first process has 2 arrivals before the second process has 1 arrival. Since each arrival is equally likely to be from the first process or the second, Prob(W) = 1/4.

(d) Find $f_{X|W}(x|W)$, the conditional probability density function for Al's time, given that he beat Bo.

$$f_{XY|W}(x,y|W) = \frac{f_{XY}(x,y)}{1/4} \text{ over } W; \quad = 0 \text{ o.w.}$$
$$\rightarrow f_{XY|W}(x,y|W) = \lambda^3 x \exp(-\lambda(x+y)) \cdot 4; \; y > x > 0$$
$$f_{X|W}(x|W) = 4\int_x^\infty \lambda^3 x e^{-\lambda x} e^{-\lambda y}\, dy = 4\lambda^2 x e^{-\lambda x}\int_x^\infty \lambda e^{-\lambda y}\, dy$$
$$= 4\lambda^2 x e^{-\lambda 2x} u(x) = (2\lambda)^2 x e^{-(2\lambda)x} u(x)$$

This is the PDF for the time until the second arrival of a process of rate 2λ. The rate = rate of process 1 + rate of process 2 $= \lambda + \lambda = 2\lambda$

(e) Independent of whether he wins or not, Bo will receive N dollars in prize money, where $N = e^{-Y}$. Find $f_N(n)$, the pdf for his winnings.

$Y = -\ln N$

range of N: $[0, 1]$

$$\left|\frac{dN}{dy}\right| = \left|-\frac{1}{N}\right| = \frac{1}{N}$$

$$f_N(n) = \frac{1}{n}\lambda e^{-\lambda(-\ln n)}; \; 0 < n \le 1$$
$$= \frac{1}{n}\lambda e^{\ln n^\lambda} = \frac{1}{n}\lambda n^\lambda = \lambda n^{\lambda-1}; \; 0 < n \le 1$$